

Ultra-relativistic spinning particle and a rotating body in external fields

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We use the vector model of spinning particle to analyze the influence of spin-field coupling on the particle's trajectory in ultra-relativistic regime. The Lagrangian with minimal spin-gravity interaction yields the equations equivalent to the Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations of a rotating body. We show that they have unsatisfactory behavior in the ultra-relativistic limit. In particular, three-dimensional acceleration of the particle increases with velocity and becomes infinite in the ultra-relativistic limit. The reason is that in the equation for trajectory emerges the term which can be thought as an effective metric generated by the minimal spin-gravity coupling. Therefore we examine the non-minimal interaction through the gravimagnetic moment κ , and show that the theory with $\kappa = 1$ is free of the problems detected in MPTD-equations. Hence the non-minimally interacting theory seem more promising candidate for description of a relativistic rotating body in general relativity.

The Lagrangian for the particle in an arbitrary electromagnetic field in Minkowski space leads to generalized Frenkel and Bargmann-Michel-Telegdi equations. The particle with magnetic moment in electromagnetic field and the particle with gravimagnetic moment in gravitational field have very similar structure of equations of motion. In particular, the spin-electromagnetic coupling also produces an effective metric for the particle with anomalous magnetic moment. If we use the usual special-relativity notions for time and distance, then the critical speed, which the particle cannot exceed during its evolution in electromagnetic field, is different from the speed of light. This can be corrected assuming that the three-dimensional geometry should be defined with respect to the effective metric.

I. INTRODUCTION

The problem of a covariant description of rotational degrees of freedom has a long and fascinating history[1]-[13]. Equations of motion of a rotating body in curved background formulated usually in the multipole approach to description of the body, see [1] for the review. The first results were reported by Mathisson [2] and Papapetrou [3]. They assumed that the structure of test body can be described by a set of multipoles and have taken the approximation which involves only first two terms (the pole-dipole approximation). The equations are then derived by integration of conservation law for the energy-momentum tensor, $T^{\mu\nu}{}_{;\mu} = 0$. A manifestly covariant equations were formulated by Tulczyjew [4] and Dixon [5, 6]. In the current literature they usually appear in the form given by Dixon (the equations (6.31)-(6.33) in [5]), we will refer them as Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations. They are widely used now to account spin effects in compact binaries and rotating black holes, see [14–16] and references therein.

Concerning the equations of spinning particle in electromagnetic field, may be the best candidates are those of Frenkel [9, 10] and Bargmann, Michel and Telegdi (BMT) [11]. Here the strong restriction on possible form of semiclassical equations is that the reasonable model should be in correspondence with the Dirac equation. In this regard the vector model of spin (see below) is of interest because it yields the Frenkel equations at the classical level, and implies the Dirac equation after canonical quantization [17].

In this work we study behavior of a particle governed by these equations (as well as by some of their generalizations) in the ultra-relativistic regime. To avoid the ambiguities in the passage from Lagrangian to Hamiltonian description and vice versa, and in the choice of possible form of interaction, we start in each case from an appropriate variational problem. The vector models of spin provide one possible way to achieve this. In these models, the basic variables¹ in spin-sector are non-Grassmann vector ω^μ and its conjugated momentum π_μ . The spin-tensor is a composite quantity constructed from these variables, $S^{\mu\nu} = 2(\omega^\mu \pi^\nu - \omega^\nu \pi^\mu)$. To have a theory with right number of physical degrees of

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¹ For early attempts to build a vector model, see the review [19].

freedom for the spin, certain constraints on the eight basic variables should follow from the variational problem. It should be noted that even for the free theory in flat space, search for the variational problem represents rather non trivial task (for the earlier attempts, see [18] and the review [19]).

To explain in a few words the problem which will be under discussion, we remind that typical relativistic equations of motion have singularity at some value of a particle speed. The singularity determines behavior of the particle in ultra-relativistic limit. For instance, the standard equations of spinless particle interacting with electromagnetic field in the physical-time parametrization $x^\mu(t) = (ct, \mathbf{x}(t))$

$$\frac{d}{dt} \left(\frac{\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \right) = \frac{e}{mc^2} F^\mu{}_\nu \dot{x}^\nu, \quad (1)$$

become singular as the relativistic contraction factor vanishes, $\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = c^2 - \mathbf{v}^2 = 0$. Rewriting the equations in the form of second law of Newton we find an acceleration. For the case, the longitudinal acceleration reads $a_{||} = \mathbf{v} \mathbf{a} = \frac{e(c^2 - \mathbf{v}^2)^{\frac{3}{2}}}{mc^3} (\mathbf{E} \mathbf{v})$, that is the factor, elevated in some degree, appears on the right hand side of the equation, and thus determines the value of velocity at which the longitudinal acceleration vanishes, $a_{||} \xrightarrow{v \rightarrow c} 0$. For the present case, the singularity implies that during its evolution in the external field, the spinless particle can not exceed the speed of light c .

In the equations for spinning particle, instead of the original metric ($\eta_{\mu\nu}$ in flat and $g_{\mu\nu}$ in curved space) emerges the effective metric $G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, with spin and field-dependent contribution $h_{\mu\nu}$. This turn out to be true both for MPTD and Frenkel equations. This leads to (drastic in some cases) changes [31, 32] in behavior of spinning particle as compare with (1). The present work is devoted to detailed analysis of the behavior in ultra-relativistic regime.

We will use the following terminology. The speed v_{cr} that a particle can not exceed during its evolution in an external field is called critical speed². The observer independent scale c of special relativity is called, as usual, the speed of light.

The work is organized as follows. In Sect. II we define three-dimensional acceleration (27) of a particle in an arbitrary gravitational field. The definition guarantees that massive spinless particle propagating along four-dimensional geodesic can not exceed the speed of light. Then we obtain the expressions (45) and (46) for the acceleration implied by equation of a general form (44). They will be repeatedly used in the subsequent sections. In Sect. III we shortly review the vector model of spin and present three equivalent Lagrangians of the free theory. In Sect. IV A we obtain equations of the particle minimally interacting with gravity starting from the Lagrangian action without auxiliary variables. The variational problem leads to the theory with fixed value of spin. In Sect. IV B we present the Lagrangian which leads to the model of Hanson-Regge type [20], with unfixed spin and with a mass-spin trajectory constraint. In Sect. IV C we present the MPTD-equations in the form convenient for our analysis, and show their equivalence with those obtained in Sect. IV A. In Sect. IV D we discuss the problems arising in ultra-relativistic limit of MPTD-equations. The first problem is the discrepancy between the critical speed with the speed of light. We should note that similar observations were mentioned in a number of works. The appearance of trajectories with space-like four-velocity was remarked by Hanson and Regge in their model of spherical top in electromagnetic field [20]. Space-like trajectories of this model in gravitational fields were studied in [21, 22]. The second problem is that the transversal acceleration increases with velocity and blows up in the ultra-relativistic limit.

In his work [23], Khriplovich proposed non-minimal interaction of a rotating body through the gravimagnetic moment κ . In Sect. V A we construct the non-minimal interaction starting from the Hamiltonian variational problem, and show (Sect. V B) that the model with $\kappa = 1$ has reasonable behavior in ultra-relativistic limit. The Lagrangian with one auxiliary variable for the particle with gravimagnetic moment is constructed in Sect. V C. In Sect. VI we construct two toy models of spinless particle with critical speed different from the speed of light. In sect. VII A we analyze generalization of the Frenkel equations to the case of a particle with magnetic moment in an arbitrary electromagnetic field in Minkowski space. Here we start from the Lagrangian action with one auxiliary variable. In Sect. VII B we show that critical speed of the particle with anomalous magnetic moment is different from the speed of light, if we use the standard special-relativity notions for time and distance. In Sect. VII C we show that the equality between the two speeds can be preserved assuming that three-dimensional geometry should be defined with respect to effective metric arised due to interaction of spin with electromagnetic field. We point out that a possibility of deformed relation between proper and laboratory time in the presence of electromagnetic field was discussed before by van Holten in his model of spin [24].

² We prefer the term critical speed instead of maximum speed since v_{cr} generally is spin and field-dependent quantity, see below.

Notation. Our variables are taken in arbitrary parametrization τ , then $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$. Covariant derivative is $\nabla P^\mu = \frac{dP^\mu}{d\tau} + \Gamma_{\alpha\beta}^\mu \dot{x}^\alpha P^\beta$ and curvature is $R^\sigma_{\lambda\mu\nu} = \partial_\mu \Gamma^\sigma_{\lambda\nu} - \partial_\nu \Gamma^\sigma_{\lambda\mu} + \Gamma^\sigma_{\beta\mu} \Gamma^\beta_{\lambda\nu} - \Gamma^\sigma_{\beta\nu} \Gamma^\beta_{\lambda\mu}$. The square brackets mean antisymmetrization, $\omega^{[\mu}\pi^{\nu]} = \omega^\mu \pi^\nu - \omega^\nu \pi^\mu$. For the four-dimensional quantities we suppress the contracted indexes and use the notation $\dot{x}^\mu G_{\mu\nu} \dot{x}^\nu = \dot{x} G \dot{x}$, $N^\mu_{\nu} \dot{x}^\nu = (N\dot{x})^\mu$, $\omega^2 = g_{\mu\nu} \omega^\mu \omega^\nu$, $\mu, \nu = 0, 1, 2, 3$. Notation for the scalar functions constructed from second-rank tensors are $\theta S = \theta^{\mu\nu} S_{\mu\nu}$, $S^2 = S^{\mu\nu} S_{\mu\nu}$.

When we work in four-dimensional Minkowski space with coordinates $x^\mu = (x^0 = ct, x^i)$, we use the metric $\eta_{\mu\nu} = (-, +, +, +)$, then $\dot{x}\omega = \dot{x}^\mu \omega_\mu = -\dot{x}^0 \omega^0 + \dot{x}^i \omega^i$ and so on. Suppressing the indexes of three-dimensional quantities, we use bold letters, $v^i \gamma_{ij} a^j = \mathbf{v} \gamma \mathbf{a}$, $v^i G_{i\mu} v^\mu = \mathbf{v} G v$, $i, j = 1, 2, 3$, and so on.

Electromagnetic field:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu = (F_{0i} = -E_i, F_{ij} = \epsilon_{ijk} B_k), \\ E_i &= -\frac{1}{c} \partial_t A_i + \partial_i A_0, \quad B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} = \epsilon_{ijk} \partial_j A_k. \end{aligned} \quad (2)$$

II. THREE-DIMENSIONAL ACCELERATION OF SPINLESS PARTICLE IN GENERAL RELATIVITY

By construction of Lorentz transformations, the speed of light in special relativity is an observer-independent quantity. As we have mentioned in the Introduction, the invariant scale is closely related with the critical speed in an external field. In a curved space we need to be more careful since the three-dimensional geometry should respect the coordinate independence of the speed of light. To achieve this, we use below the Landau and Lifshitz procedure [25] to define time interval, three-dimensional distance and velocity. Then we introduce the notion of three-dimensional acceleration which guarantees that massive spinless particle propagating along four-dimensional geodesic can not exceed the speed of light. The expression (46) for longitudinal acceleration implied by equation of the form (44) will be repeatedly used in subsequent sections.

Consider an observer that labels the events by the coordinates x^μ of pseudo Riemann space [25, 26]

$$\mathbf{M}^{(1,3)} = \{x^\mu, g_{\mu\nu}(x^\rho), \text{sign } g_{\mu\nu} = (-, +, +, +)\}, \quad (3)$$

to describe the motion of a particle in gravitational field with metric $g_{\mu\nu}$. Formal definitions of three-dimensional quantities subject to the discussion can be obtained representing interval in 1 + 3 block-diagonal form [25]

$$\begin{aligned} -ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = \\ -c^2 \left[\frac{\sqrt{-g_{00}}}{c} (dx^0 + \frac{g_{0i}}{g_{00}} dx^i) \right]^2 &+ \left(g_{ij} - \frac{g_{0i} g_{0j}}{g_{00}} \right) dx^i dx^j. \end{aligned} \quad (4)$$

This prompts to introduce infinitesimal time interval, distance and speed as follows:

$$dt = \frac{\sqrt{-g_{00}}}{c} (dx^0 + \frac{g_{0i}}{g_{00}} dx^i) \equiv -\frac{g_{0\mu} dx^\mu}{c\sqrt{-g_{00}}}. \quad (5)$$

$$dl^2 = (g_{ij} - \frac{g_{0i} g_{0j}}{g_{00}}) dx^i dx^j \equiv \gamma_{ij} dx^i dx^j, \quad v = \frac{dl}{dt}. \quad (6)$$

Therefore the conversion factor between intervals of the world time $\frac{dx^0}{c}$ and the time dt measured by laboratory clock is

$$\frac{dt}{dx^0} = \frac{\sqrt{-g_{00}}}{c} (1 + \frac{g_{0i}}{g_{00}} \frac{dx^i}{dx^0}). \quad (7)$$

Introduce also the three-velocity vector \mathbf{v} with components

$$v^i = \left(\frac{dt}{dx^0} \right)^{-1} \frac{dx^i}{dx^0}, \quad (8)$$

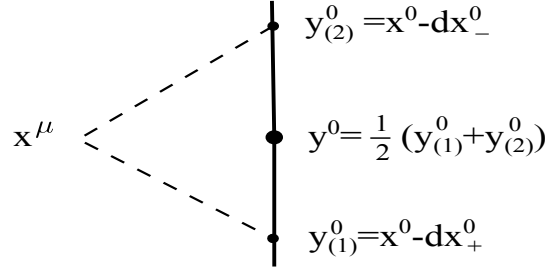


FIG. 1: Definition of simultaneous events. The vertical line represents a world-line of the laboratory clock. The points $y_{(1)}^0$ and $y_{(2)}^0$ have null-interval with x^μ . Then the middle point y^0 represents the event simultaneous with x^μ .

or, symbolically, $v^i = \frac{dx^i}{dt}$. We stress that contrary to $\frac{d}{dx^\mu}$, the set $(\frac{d}{dt}, \frac{d}{dx^i})$ is non-holonomic basis of tangent space³. This does not represent any special problem for our discussion since we are interested in the differential quantities such as velocity and acceleration.

The equation (8) is consistent with the above definition of v : $v^2 = (\frac{dl}{dt})^2 = \mathbf{v}^2 = v^i \gamma_{ij} v^j$. In the result, the interval acquires the form similar to special relativity (but now we have $\mathbf{v}^2 = \mathbf{v} \gamma \mathbf{v}$)

$$-ds^2 = -c^2 dt^2 + dl^2 = -c^2 dt^2 \left(1 - \frac{\mathbf{v}^2}{c^2}\right). \quad (9)$$

This equality holds in any coordinate system x^μ . Hence a particle with the propagation law $ds^2 = 0$ has the speed $\mathbf{v}^2 = c^2$, and this is a coordinate-independent statement.

For the latter use we also introduce the four-dimensional quantity

$$v^\mu = \left(\frac{dt}{dx^0}\right)^{-1} \frac{dx^\mu}{dx^0} = \left(\left(\frac{dt}{dx^0}\right)^{-1}, \mathbf{v}\right). \quad (10)$$

Combining the equations (8) and (7), we can present the conversion factor in terms of three-velocity as follows:

$$\left(\frac{dt}{dx^0}\right)^{-1} = v^0 = \frac{c}{\sqrt{-g_{00}}} - \frac{g_{0i} v^i}{g_{00}}. \quad (11)$$

These rather formal tricks are based [25] on the notion of simultaneity in general relativity and on the analysis of flat limit. Four-interval of special relativity has direct physical interpretation in two cases. First, for two events which occur at the same point, the four-interval is proportional to time interval, $dt = -\frac{ds}{c}$. Second, for simultaneous events the four-interval coincides with distance, $dl = ds$. Assuming that the same holds in general relativity, let us analyze infinitesimal time interval and distance between two events with coordinates x^μ and $x^\mu + dx^\mu$. The world line $y^\mu = (y^0, \mathbf{y} = \text{const})$ is associated with laboratory clock placed at the spacial point \mathbf{y} . So the time-interval between the events (y^0, \mathbf{y}) and $(y^0 + dy^0, \mathbf{y})$ measured by the clock is

$$dt = -\frac{ds}{c} = \frac{\sqrt{-g_{00}}}{c} dy^0. \quad (12)$$

Consider the event x^μ infinitesimally closed to the world line $(y^0, \mathbf{y} = \text{const})$. To find the event on the world line which is simultaneous with x^μ , we first look for the events $y_{(1)}^\mu$ and $y_{(2)}^\mu$ which have null-interval with x^μ , $ds(x^\mu, y_{(a)}^\mu) = 0$. The equation $g_{\mu\nu} dx^\mu dx^\nu = 0$ with $dx^\mu = x^\mu - y^\mu$ has two solutions $dx_\pm^0 = \frac{g_{0i} dx^i}{-g_{00}} \pm \frac{\sqrt{d\mathbf{x}\gamma d\mathbf{x}}}{\sqrt{-g_{00}}}$, then $y_{(1)}^0 = x^0 - dx_+^0$ and

³ Let $e_\mu = \tilde{a}^\alpha_\mu \partial_\alpha$ be a basis of tangent space and $e^\mu = a^\mu_\alpha dx^\alpha$, where $a^\mu_\alpha \tilde{a}^\alpha_\nu = \delta^\mu_\nu$, be the dual basis for e_μ , that is $e^\mu(e_\nu) = \delta^\mu_\nu$. e_μ is the holonomic basis (that is $e_\mu = \frac{\partial}{\partial x^\mu}$ are tangent to some coordinate lines x^μ) if $(e_\mu e_\nu - e_\nu e_\mu)f = 0$. For the matrix a^μ_α which determines the dual basis e^μ , this condition reduces to the simple equation $\partial_\mu a^\alpha_\nu - \partial_\nu a^\alpha_\mu = 0$. For the matrix which determines our 1+3 decomposition we have $a^0_\mu = -\frac{g_{0\mu}}{c\sqrt{-g_{00}}}$, $a^i_0 = 0$, and $a^i_j = \delta^i_j$, then, for instance, $\partial_\mu a^0_\nu - \partial_\nu a^0_\mu = -\frac{1}{c\sqrt{-g_{00}}}(\partial_\mu g_{0\nu} - \partial_\nu g_{0\mu}) \neq 0$. So the set $(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i})$ generally does not represent a holonomic basis.

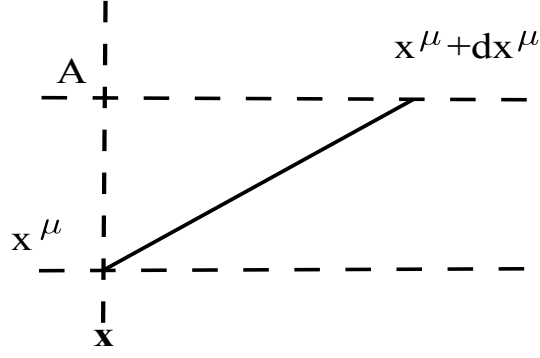


FIG. 2: Definition of simultaneous events. The vertical line represents a world-line of the laboratory clock. The points $y_{(1)}^0$ and $y_{(2)}^0$ have null-interval with x^μ . Then the middle point y^0 represents the event simultaneous with x^μ .

$y_{(2)}^0 = x^0 - dx_-^0$. Second, we compute the middle point

$$y^0 = \frac{1}{2}(y_{(1)}^0 + y_{(2)}^0) = x^0 + \frac{g_{0i}dx^i}{g_{00}}. \quad (13)$$

By definition⁴, the event (y^0, \mathbf{y}) with the null-coordinate (13) is simultaneous with the event (x^0, \mathbf{x}) , see Figure 1. By this way we synchronized clocks at the spacial points \mathbf{x} and \mathbf{y} . According to (13), the simultaneous events have different null-coordinates, and the difference dx^0 obeys the equation

$$dx^0 + \frac{g_{0i}dx^i}{g_{00}} = 0. \quad (14)$$

Consider a particle which propagated from x^μ to $x^\mu + dx^\mu$. Let us compute time-interval and distance between these two events. According to (13), the event

$$\left(x^0 + dx^0 + \frac{g_{0i}dx^i}{g_{00}}, \mathbf{x} \right), \quad (15)$$

at the spacial point \mathbf{x} is simultaneous with $x^\mu + dx^\mu$, see Figure 2. According to (12) and (13), the time interval between the events x^μ and (15) is

$$dt = \frac{\sqrt{-g_{00}}}{c} \left(dx^0 + \frac{g_{0i}dx^i}{g_{00}} \right). \quad (16)$$

Since the events $x^\mu + dx^\mu$ and (15) are simultaneous, this equation gives also the time interval between x^μ and $x^\mu + dx^\mu$. Further, the difference of coordinates between the events $x^\mu + dx^\mu$ and (15) is $dz^\mu = \left(-\frac{g_{0i}dx^i}{g_{00}}, dx^i \right)$. As they are simultaneous, the distance between them is

$$dl^2 = -ds^2 = g_{\mu\nu}dz^\mu dz^\nu = (g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}})dx^i dx^j \equiv \gamma_{ij}dx^i dx^j. \quad (17)$$

Since (15) occur at the same spacial point as x^μ , this equation gives also the distance between x^μ and $x^\mu + dx^\mu$. The equations (16) and (17) coincide with the formal definitions presented above, Eqs. (5) and (6).

We now turn to the definition of three-acceleration. The spinless particle in general relativity follows a geodesic line. If we take the proper time to be the parameter, geodesics obey the system

$$\nabla_s \frac{dx^\mu}{ds} \equiv \frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0, \quad g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = -1, \quad (18)$$

⁴ In the flat limit the sequence $y_{(1)}^\mu, x^\mu, y_{(2)}^\mu$ of events can be associated with emission, reflection and absorption of a photon with the propagation law $ds = 0$. Then the middle point (13) should be considered simultaneous with x^0 .

where

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}). \quad (19)$$

Due to this definition, the system (18) obeys the identity $g_{\mu\nu} \frac{dx^\mu}{ds} \nabla_s \frac{dx^\nu}{ds} = 0$.

The system in this parametrization has no sense for the case we are interested in, $ds^2 \rightarrow 0$. So we rewrite it in arbitrary parametrization τ

$$\frac{d\tau}{ds} \frac{d}{d\tau} \left(\frac{d\tau}{ds} \frac{dx^\mu}{d\tau} \right) + \left(\frac{d\tau}{ds} \right)^2 \Gamma^\mu_{\alpha\beta}(g) \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0, \quad \frac{d\tau}{ds} = \frac{1}{\sqrt{-\dot{x}g\dot{x}}}, \quad (20)$$

this yields the equation of geodesic line in reparametrization-invariant form

$$\frac{1}{\sqrt{-\dot{x}g\dot{x}}} \frac{d}{d\tau} \left(\frac{\dot{x}^\mu}{\sqrt{-\dot{x}g\dot{x}}} \right) = -\Gamma^\mu_{\alpha\beta}(g) \frac{\dot{x}^\alpha}{\sqrt{-\dot{x}g\dot{x}}} \frac{\dot{x}^\beta}{\sqrt{-\dot{x}g\dot{x}}}. \quad (21)$$

The formalism (5)-(9) remains manifestly covariant under subgroup of spacial transformations $x^0 = x'^0$, $x^i = x'^i(x'^j)$, $\frac{\partial x^i}{\partial x'^j} \equiv a^i_j(x')$. Under these transformations g_{00} is a scalar function, g_{0i} is a vector while g_{ij} and γ_{ij} are tensors. Since $g^{ij}\gamma_{jk} = \delta^i_k$, the inverse metric of γ_{ij} turns out to be $(\gamma^{-1})^{ij} = g^{ij}$. Introduce the covariant derivatives ∇_k of a vector field $\xi^i(x^0, x^k)$

$$\nabla_k \xi^i = \partial_k \xi^i + \tilde{\Gamma}^i_{kj}(\gamma) \xi^j. \quad (22)$$

The three-dimensional Christoffel symbols $\tilde{\Gamma}^i_{jk}(\gamma)$ are constructed with help of three-dimensional metric $\gamma_{ij}(x^0, x^k)$ written in Eq. (6), where x^0 is considered as a parameter

$$\tilde{\Gamma}^i_{jk}(\gamma) = \frac{1}{2}\gamma^{ia}(\partial_j \gamma_{ak} + \partial_k \gamma_{aj} - \partial_a \gamma_{jk}). \quad (23)$$

As a consequence, the metric γ is covariantly constant, $\nabla_k \gamma_{ij} = 0$.

The velocity (8) behaves as a vector, $v^i(x^0) = a^i_j(x'^k(x^0))v'^j(x^0)$, so below we use also the covariant derivative

$$\nabla_0 v^i = \frac{dv^i}{dx^0} + \tilde{\Gamma}^i_{jk}(\gamma) \frac{dx^j}{dx^0} v^k. \quad (24)$$

We associated with $\mathbf{M}^{(1,3)}$ the one-parameter family of three-dimensional spaces $\mathbf{M}^3_{x^0} = \{x^k, \gamma_{ij}, \nabla_k \gamma_{ij} = 0\}$. Note that velocity has been defined above as a tangent vector to the curve which cross the family and is parameterized by this parameter, $x^i(x^0)$.

To define an acceleration of a particle in the three-dimensional geometry, we need the notion of a constant vector field (or, equivalently, the parallel-transport equation). In the case of stationary field, $g_{\mu\nu}(x^k)$, we can identify the curve $x^i(x^0)$ of $\mathbf{M}^{(1,3)}$ with that of any one of $\mathbf{M}^3_{x^0} = \{x^k, \gamma_{ij}(x^k)\}$. So we have the usual three-dimensional Riemann geometry, and an analog of constant vector field of Euclidean geometry is the covariantly-constant field along the line $x^i(x^0)$, $\nabla_0 \xi^i = 0$. For the field of velocity, its deviation from the covariant constancy is the acceleration [32]

$$a^i = \left(\frac{dt}{dx^0} \right)^{-1} \nabla_0 v^i = \left(\frac{dt}{dx^0} \right)^{-1} \frac{dv^i}{dx^0} + \tilde{\Gamma}^i_{jk} v^j v^k. \quad (25)$$

To define an acceleration in general case, $\gamma_{ij}(x^0, x^i)$, we need to adopt some notion of a constant vector field along the trajectory $x^i(x^0)$ that cross the family $\mathbf{M}^3_{x^0}$. We propose the definition which preserves one of basic properties of constant fields in differential geometry. In Euclidean and Minkowski spaces, the canonical scalar product of two constant fields does not depend on the point where it was computed. In (pseudo) Riemann space, constant vector field is defined in such a way that the same property holds [37]. In particular, taking the scalar product along a line $x^i(x^0)$, we have $\frac{d}{dx^0}(\xi, \eta) = 0$. For the constant fields in the three-dimensional geometry resulting after Landau-Lifshitz 1 + 3-decomposition, we demand the same (necessary) condition: $\frac{d}{dx^0}[\xi^i(x^0)\gamma_{ij}(x^0, x^i(x^0))\eta^j(x^0)] = 0$. Taking into account that $\nabla_k \gamma_{ij} = 0$, this condition can be written as follows

$$(\nabla_0 \xi + \frac{1}{2}\xi \partial_0 \gamma \gamma^{-1}, \eta) + (\xi, \nabla_0 \eta + \frac{1}{2}\gamma^{-1} \partial_0 \gamma \eta) = 0.$$

This equation is satisfied, if we take the parallel-transport equation to be

$$\nabla_0 \xi^i + \frac{1}{2} (\xi \partial_0 \gamma \gamma^{-1})^i = 0. \quad (26)$$

Deviation from the constant field is an acceleration. So we define acceleration with respect to physical time as follows:

$$a^i = \left(\frac{dt}{dx^0} \right)^{-1} \left[\nabla_0 v^i + \frac{1}{2} (\mathbf{v} \partial_0 \gamma \gamma^{-1})^i \right]. \quad (27)$$

For the special case of stationary field, $g_{\mu\nu}(x^i)$, the definition (27) reduces to (25) and to that of Landau and Lifshitz, see page 251 in [25].

The extra-term appeared in this equation plays an essential role to provide that for the geodesic motion we have: $a_{||} \xrightarrow{v \rightarrow c} 0$. As a consequence, geodesic particle in gravitational field can not exceed the speed of light. To show this, we compute the longitudinal acceleration ($\mathbf{v} \gamma \mathbf{a}$) implied by geodesic equation (21). Take $\tau = x^0$, then $\sqrt{-\dot{x} g \dot{x}} = \frac{dt}{dx^0} \sqrt{c^2 - \mathbf{v} \gamma \mathbf{v}}$, and spacial part of (21) is

$$\left(\frac{dt}{dx^0} \right)^{-1} \frac{d}{dx^0} \frac{v^i}{\sqrt{c^2 - \mathbf{v} \gamma \mathbf{v}}} = \frac{f^i}{\sqrt{c^2 - \mathbf{v} \gamma \mathbf{v}}}, \quad (28)$$

where

$$f^i(v^\mu) = - \left(\frac{dt}{dx^0} \right)^{-2} \Gamma^i_{00} - \Gamma^i_{jk} v^j v^k - 2 \left(\frac{dt}{dx^0} \right)^{-1} \Gamma^i_{0k} v^k = -\Gamma^i_{\mu\nu} v^\mu v^\nu, \quad (29)$$

is non-singular function as $v \rightarrow c$. Computing derivative on the l.h.s. of (28), we complete $\frac{dv^i}{dx^0}$ up to covariant derivative $\nabla_0 v^i$

$$\frac{d}{dx^0} \frac{v^i}{\sqrt{c^2 - \mathbf{v} \gamma \mathbf{v}}} = \nabla_0 v^i - \tilde{\Gamma}^i_{jk}(\gamma) v^j v^k \frac{dt}{dx^0} + \frac{v^i}{2(c^2 - \mathbf{v} \gamma \mathbf{v})} \frac{d}{dx^0} (\mathbf{v} \gamma \mathbf{v}). \quad (30)$$

For the derivative contained in the last term we find, using covariant constancy of γ

$$\frac{d}{dx^0} [\mathbf{v} \gamma (x^0, x^i) \mathbf{v}] = 2\mathbf{v} \gamma \nabla_0 \mathbf{v} + \mathbf{v} \partial_0 \gamma \mathbf{v} + \mathbf{v} \nabla_0 \gamma \mathbf{v} = 2\mathbf{v} \gamma \nabla_0 \mathbf{v} + \mathbf{v} \partial_0 \gamma \mathbf{v}. \quad (31)$$

Then (28) acquires the form

$$\left(\frac{dt}{dx^0} \right)^{-1} \left[M^i_j \nabla_0 v^j + \frac{(\mathbf{v} \partial_0 \gamma \mathbf{v})}{2(c^2 - \mathbf{v} \gamma \mathbf{v})} v^i \right] = f^i + \tilde{\Gamma}^i_{kl} v^k v^l, \quad (32)$$

where

$$M^i_j = \delta^i_j + \frac{v^i (\mathbf{v} \gamma)_j}{c^2 - \mathbf{v} \gamma \mathbf{v}}. \quad (33)$$

We apply the inverse matrix

$$\tilde{M}^i_j = \delta^i_j - \frac{v^i (\mathbf{v} \gamma)_j}{c^2}, \quad (34)$$

and use the identity

$$\tilde{M}^i_j v^j = \frac{c^2 - \mathbf{v} \gamma \mathbf{v}}{c^2} v^i, \quad (35)$$

then

$$\left(\frac{dt}{dx^0} \right)^{-1} \left[\nabla_0 v^i + \frac{(\mathbf{v} \partial_0 \gamma \mathbf{v})}{2c^2} v^i \right] = \tilde{M}^i_j \left[f^j + \tilde{\Gamma}^j_{kl} v^k v^l \right]. \quad (36)$$

Next, we complete $\nabla_0 v^i$ up to the acceleration (27). Then (36) yields

$$a^i = \frac{1}{2} \left(\frac{dt}{dx^0} \right)^{-1} \left[(\mathbf{v} \partial_0 \gamma \gamma^{-1})^i - \frac{(\mathbf{v} \partial_0 \gamma \mathbf{v})}{c^2} v^i \right] + \tilde{M}^i_j [-\Gamma^j_{\mu\nu} v^\mu v^\nu + \tilde{\Gamma}^j_{kl}(\gamma) v^k v^l]. \quad (37)$$

Contracting this with $(\mathbf{v}\gamma)_i$, we use $(\mathbf{v}\gamma)_i \tilde{M}^i_j = \frac{c^2 - \mathbf{v}\gamma\mathbf{v}}{c^2} (\mathbf{v}\gamma)_j$ and obtain longitudinal acceleration

$$\mathbf{v}\gamma\mathbf{a} = \frac{1}{2} \left(\frac{dt}{dx^0} \right)^{-1} \left[(\mathbf{v} \partial_0 \gamma \mathbf{v}) - (\mathbf{v} \partial_0 \gamma \mathbf{v}) \frac{(\mathbf{v}\gamma\mathbf{v})}{c^2} \right] + \left(1 - \frac{\mathbf{v}\gamma\mathbf{v}}{c^2} \right) (\mathbf{v}\gamma)_i [-\Gamma^i_{\mu\nu} v^\mu v^\nu + \tilde{\Gamma}^i_{kl}(\gamma) v^k v^l]. \quad (38)$$

This implies $\mathbf{v}\gamma\mathbf{a} \rightarrow 0$ as $\mathbf{v}\gamma\mathbf{v} \rightarrow c^2$.

The last term in (27) yields the important factor $(\mathbf{v} \partial_0 \gamma \mathbf{v})$ in Eq. (38). As the equations of motion (37) and (38) do not contain the square root $\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}$, they have sense even for $v > c$. Without this factor, we would have $\mathbf{v}\gamma\mathbf{a} \neq 0$ as $\mathbf{v}\gamma\mathbf{v} \rightarrow c^2$, so the particle in gravitational field could exceed c and then continues accelerate. The same happen if we try to define acceleration using usual derivative instead of the covariant one⁵.

Let us confirm that c is the only special point of the function (38) representing the longitudinal acceleration. Using Eqs. (19), (6)-(10), (23) and the identities

$$\gamma_{ij} g^{jk} = \delta_i^k, \quad \gamma_{ij} g^{j0} = -\frac{g_{0i}}{g_{00}}, \quad (39)$$

we can present the right hand side of Eq. (38) in terms of initial metric as follows

$$\begin{aligned} \mathbf{v}\gamma\mathbf{a} = \frac{c^2 - \mathbf{v}\gamma\mathbf{v}}{2c\sqrt{-g_{00}}} & \left\{ \frac{c}{\sqrt{-g_{00}}} \left[\left(\frac{dt}{dx^0} \right)^{-1} \partial_0 g_{00} + v^k \partial_k g_{00} \right] - \partial_0 g_{00} \left(\frac{dt}{dx^0} \right)^{-2} - 2\partial_0 g_{0k} \left(\frac{dt}{dx^0} \right)^{-1} v^k - \partial_0 g_{kl} v^k v^l \right\} \equiv \\ & \frac{c^2 - \mathbf{v}\gamma\mathbf{v}}{2c\sqrt{-g_{00}}} \left\{ \frac{c}{\sqrt{-g_{00}}} v^\mu \partial_\mu g_{00} - \partial_0 g_{\mu\nu} v^\mu v^\nu \right\}. \end{aligned} \quad (40)$$

The quantity v^μ has been defined in (10). Excluding v^0 according to this expression, we obtain

$$\mathbf{v}\gamma\mathbf{a} = \frac{c^2 - \mathbf{v}\gamma\mathbf{v}}{2\sqrt{-g_{00}}} \left\{ \frac{v^k \partial_k g_{00}}{\sqrt{-g_{00}}} - 2\partial_0 \left(\frac{g_{0i}}{\sqrt{-g_{00}}} \right) v^i - \frac{1}{c} \partial_0 \gamma_{ij} v^i v^j \right\}. \quad (41)$$

For the stationary metric, $g_{\mu\nu}(x^k)$, the equation (41) acquires a specially simple form

$$\mathbf{v}\gamma\mathbf{a} = -(c^2 - \mathbf{v}\gamma\mathbf{v}) \frac{v^k \partial_k g_{00}}{2g_{00}}. \quad (42)$$

This shows that the longitudinal acceleration has only one special point in the stationary gravitational field, $\mathbf{v}\gamma\mathbf{a} \rightarrow 0$ as $\mathbf{v}\gamma\mathbf{v} \rightarrow c^2$. Then the same is true in general case (40), at least for the metric which is sufficiently slowly varied in time.

While we have discussed the geodesic equation, the computation which leads to the formula (38) can be repeated for a more general equation. Let us formulate the result which will be repeatedly used below. Using the factor $\sqrt{-\dot{x}g\dot{x}}$ we construct the reparametrization-invariant derivative

$$D = \frac{1}{\sqrt{-\dot{x}g\dot{x}}} \frac{d}{d\tau}. \quad (43)$$

Consider the reparametrization-invariant equation of the form

$$DDx^\mu(\tau) = \mathcal{F}^\mu(Dx^\nu, \dots). \quad (44)$$

⁵ Indeed, instead of Eq. (27), let us define an acceleration according to the expression $a^i = \left(\frac{dt}{dx^0} \right)^{-1} \left[\frac{dv^i}{dx^0} + \frac{1}{2} (\mathbf{v} \partial_0 \gamma \gamma^{-1})^i \right]$. Then for the geodesic particle we obtain, instead of (38), the longitudinal acceleration $\mathbf{v}\gamma\mathbf{a}' = [\text{r.h.s. of (38)}] - (\mathbf{v}\gamma)_i \tilde{\Gamma}^i_{jk}(\gamma) \frac{dx^j}{dx^0} v^k = [\text{r.h.s. of (38)}] - \frac{1}{2} \partial_i \gamma_{jk} v^i v^j v^k$. The extra term does not involve the factor $c^2 - \mathbf{v}\gamma\mathbf{v}$ and so does not vanish at $|v| = c$.

and suppose that the three-dimensional geometry is defined by $g_{\mu\nu}$. Then Eq. (44) implies the three-acceleration

$$a^i = \tilde{M}^i_j \left[(c^2 - \mathbf{v}\gamma\mathbf{v})\mathcal{F}^j + \tilde{\Gamma}^j_{kl}(\gamma)v^k v^l \right] + \frac{1}{2} \left(\frac{dt}{dx^0} \right)^{-1} \left[(\mathbf{v}\partial_0\gamma\gamma^{-1})^i - \frac{v^i}{c^2}(\mathbf{v}\partial_0\gamma\mathbf{v}) \right], \quad (45)$$

and the longitudinal acceleration

$$\mathbf{v}\gamma\mathbf{a} = \frac{(c^2 - \mathbf{v}\gamma\mathbf{v})^2}{c^2}(\mathbf{v}\gamma\mathcal{F}) + \frac{c^2 - \mathbf{v}\gamma\mathbf{v}}{c^2} \left[(\mathbf{v}\gamma)_i \tilde{\Gamma}^i_{kl}(\gamma)v^k v^l + \frac{1}{2} \left(\frac{dt}{dx^0} \right)^{-1} (\mathbf{v}\partial_0\gamma\mathbf{v}) \right]. \quad (46)$$

The spacial part of the force is $\mathcal{F}^i = \mathcal{F}^i(\frac{v^\nu}{\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}})$, where v^μ is given by (10), and the connection $\tilde{\Gamma}^i_{kl}(\gamma)$ is constructed with help of the three-dimensional metric $\gamma_{ij} = (g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}})$ according to (23). For the geodesic equation in this notation we have $\mathcal{F}^i = -\Gamma^i_{\mu\nu} \frac{v^\mu v^\nu}{c^2 - \mathbf{v}\gamma\mathbf{v}}$. With this \mathcal{F}^i the equations (45) and (46) coincide with (37) and (38).

III. VECTOR MODEL OF RELATIVISTIC SPIN

The variational problem for vector model of spin interacting with electromagnetic and gravitational fields can be formulated with various sets of auxiliary variables [17, 27–30]. For the free theory in flat space there is the Lagrangian action without auxiliary variables. Configuration space consist of the position $x^\mu(\tau)$ and non Grassmann vector $\omega^\mu(\tau)$ attached to the point x^μ . The action reads [29, 31]

$$S = -\frac{1}{\sqrt{2}} \int d\tau \sqrt{m^2 c^2 - \frac{\alpha}{\omega^2}} \sqrt{-\dot{x}N\dot{x} - \dot{\omega}N\dot{\omega} + \sqrt{[\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega}]^2 - 4(\dot{x}N\dot{\omega})^2}}. \quad (47)$$

The matrix $N_{\mu\nu}$ is the projector on the plane orthogonal to ω^ν

$$N_{\mu\nu} = \eta_{\mu\nu} - \frac{\omega_\mu \omega_\nu}{\omega^2}, \quad \text{then} \quad N_{\mu\alpha} N^{\alpha\nu} = N_\mu{}^\nu, \quad N_{\mu\nu} \omega^\nu = 0. \quad (48)$$

The double square-root structure in the expression (47) seem to be typical for the vector models of spin [20, 33]. This yields the primary constraint T_4 in (61) and, at the end, the supplementary spin condition (77). The parameter m is mass, while α determines the value of spin. The value $\alpha = \frac{3\hbar^2}{4}$ corresponds to an elementary spin one-half particle. The model is invariant⁶ under reparametrizations and local spin-plane symmetries [34].

The spin is described by Frenkel spin-tensor [9]. In our model this is a composite quantity constructed from ω^μ and its conjugated momentum $\pi_\mu = \frac{\partial L}{\partial \dot{\omega}^\mu}$ as follows:

$$S^{\mu\nu} = 2(\omega^\mu \pi^\nu - \omega^\nu \pi^\mu) = (S^{i0} = D^i, \quad S_{ij} = 2\epsilon_{ijk} S_k), \quad (49)$$

then $S_i = \epsilon_{ijk} \omega_j \pi_k = \frac{1}{4} \epsilon_{ijk} S_{jk}$. Here S_i is three-dimensional spin-vector and D_i is dipole electric moment [12]. In contrast to its constituents ω^μ and π^ν , the spin-tensor is invariant under local spin-plane symmetry, and thus represents an observable quantity. Canonical quantization of the model yields one-particle sector of the Dirac equation [17].

In the formulation (47), the model admits minimal interaction with electromagnetic field and with gravity. This does not spoil the number and the algebraic structure of constraints presented in the free theory. To describe the spinning particle with magnetic and gravimagnetic moments, we will need the following two reformulations.

In the spinless limit, $\alpha = 0$ and $\omega^\mu = 0$, the functional (47) reduces to the standard expression, $-mc\sqrt{-\dot{x}^\mu \dot{x}_\mu}$. The latter can be written in equivalent form using the auxiliary variable $\lambda(\tau)$ as follows: $\frac{1}{2\lambda} \dot{x}^2 - \frac{\lambda}{2} m^2 c^2$. Similarly to this, (47) can be presented in the equivalent form

$$L = \frac{1}{4\lambda_1} \left[\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega} - \sqrt{[\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega}]^2 - 4(\dot{x}N\dot{\omega})^2} \right] - \frac{\lambda_1}{2} [(mc)^2 - \frac{\alpha}{\omega^2}]. \quad (50)$$

⁶ The reparametrizations are $\tau \rightarrow \tau'(\tau)$, $x'^\mu(\tau') = x^\mu(\tau)$, $\omega'^\mu(\tau') = \omega^\mu(\tau)$, that is both x and ω are scalar functions. The local spin-plane transformations act in the plane determined by the vectors ω^μ and π^ν , see [34].

In this formulation our model admits interaction of spin with an arbitrary electromagnetic field through the magnetic moment, see section VII A. Another form of the Lagrangian is

$$L = -\sqrt{(mc)^2 - \frac{\alpha}{\omega^2}} \sqrt{(1 - \lambda^2)^{-1} [-\dot{x}N\dot{x} - \dot{\omega}N\dot{\omega} + 2\lambda\dot{x}N\dot{\omega}]} \equiv \quad (51)$$

$$-\sqrt{(mc)^2 - \frac{\alpha}{\omega^2}} \sqrt{-(N\dot{x}, N\dot{\omega}) \begin{pmatrix} \frac{\eta}{1-\lambda^2} & \frac{-\lambda\eta}{1-\lambda^2} \\ \frac{-\lambda\eta}{1-\lambda^2} & \frac{\eta}{1-\lambda^2} \end{pmatrix} \begin{pmatrix} N\dot{x} \\ N\dot{\omega} \end{pmatrix}}. \quad (52)$$

Its advantage is that the expression under the square root represents quadratic form with respect the velocities \dot{x} and $\dot{\omega}$. To relate the Lagrangians (47) and (51), we exclude λ from the latter. Computing variation of (51) with respect to λ , we obtain the equation

$$(\dot{x}N\dot{\omega})\lambda^2 - (\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega})\lambda + (\dot{x}N\dot{\omega}) = 0, \quad (53)$$

which determines λ

$$\lambda_{\pm} = \frac{(\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega}) \pm \sqrt{(\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega})^2 - 4(\dot{x}N\dot{\omega})^2}}{2(\dot{x}N\dot{\omega})}. \quad (54)$$

We substitute λ_+ into (51) and use $\lambda_+\lambda_- = 1$, then (51) turns into (47). In the formulation (51) our model admits interaction of spin with gravity through the gravimagnetic moment, see section V C.

IV. MINIMAL INTERACTION WITH AN ARBITRARY GRAVITATIONAL FIELD

A. Lagrangian and Hamiltonian formulations

The minimal interaction with gravitational field can be achieved by covariantization of the formulation without auxiliary variables. In the expressions (47) and (48) we replace $\eta_{\mu\nu} \rightarrow g_{\mu\nu}$, and usual derivative by the covariant one, $\dot{\omega}^\mu \rightarrow \nabla\omega^\mu = \frac{d\omega^\mu}{d\tau} + \Gamma_{\alpha\beta}^\mu \dot{x}^\alpha \omega^\beta$. Thus our Lagrangian in a curved background reads [30]

$$\begin{aligned} L &= -\frac{1}{\sqrt{2}} \left[m^2 c^2 - \frac{\alpha}{\omega^2} \right]^{\frac{1}{2}} \sqrt{-\dot{x}N\dot{x} - \nabla\omega N\nabla\omega + \sqrt{[\dot{x}N\dot{x} + \nabla\omega N\nabla\omega]^2 - 4(\dot{x}N\nabla\omega)^2}} \\ &\equiv -\frac{1}{\sqrt{2}} \left[m^2 c^2 - \frac{\alpha}{\omega^2} \right]^{\frac{1}{2}} L_0. \end{aligned} \quad (55)$$

Velocities \dot{x}^μ , $\nabla\omega^\mu$ and projector $N_{\mu\nu}$ transform like contravariant vectors and covariant tensor, so the action is manifestly invariant under general-coordinate transformations.

Let us construct Hamiltonian formulation of the model (55). Conjugate momenta for x^μ and ω^μ are $p_\mu = \frac{\partial L}{\partial \dot{x}^\mu}$ and $\pi_\mu = \frac{\partial L}{\partial \dot{\omega}^\mu}$ respectively. Due to the presence of Christoffel symbols in $\nabla\omega^\mu$, the conjugated momentum p_μ does not transform as a vector, so it is convenient to introduce the canonical momentum

$$P_\mu \equiv p_\mu - \Gamma_{\alpha\mu}^\beta \omega^\alpha \pi_\beta, \quad (56)$$

the latter transforms as a vector under general transformations of coordinates. Manifest form of the momenta is as follows:

$$P_\mu = \frac{1}{\sqrt{2}L_0} \left[m^2 c^2 - \frac{\alpha}{\omega^2} \right]^{\frac{1}{2}} [N_{\mu\nu} \dot{x}^\nu - K_\mu], \quad (57)$$

$$\pi_\mu = \frac{1}{\sqrt{2}L_0} \left[m^2 c^2 - \frac{\alpha}{\omega^2} \right]^{\frac{1}{2}} [N_{\mu\nu} \nabla\omega^\nu - R_\mu], \quad (58)$$

with

$$\begin{aligned} K_\mu &= T^{-1/2} [(\dot{x}N\dot{x} + \nabla\omega N\nabla\omega)(N\dot{x})_\mu - 2(\dot{x}N\nabla\omega)(N\nabla\omega)_\mu], \\ R_\mu &= T^{-1/2} [(\dot{x}N\dot{x} + \nabla\omega N\nabla\omega)(N\nabla\omega)_\mu - 2(\dot{x}N\nabla\omega)(N\dot{x})_\mu]. \end{aligned}$$

These vectors obey the following remarkable identities

$$\begin{aligned} K^2 &= \dot{x}N\dot{x}, \quad R^2 = \nabla\omega N\nabla\omega, \quad KR = -\dot{x}N\nabla\omega, \quad \dot{x}R + \nabla\omega K = 0, \\ K\dot{x} + R\nabla\omega &= \sqrt{[\dot{x}N\dot{x} + \nabla\omega N\nabla\omega]^2 - 4(\dot{x}N\nabla\omega)^2}. \end{aligned} \quad (59)$$

Using (48) we conclude that $\omega\pi = 0$ and $P\omega = 0$, that is we found two primary constraints. Using the relations (59) we find one more primary constraint, $P\pi = 0$. At last, computing $P^2 + \pi^2$ given by (57) and (58) we see that all the terms with derivatives vanish, and we obtain the primary constraint

$$T_1 \equiv P^2 + m^2c^2 + \pi^2 - \frac{\alpha}{\omega^2} = 0. \quad (60)$$

In the result, the action (55) implies four primary constraints, T_1 and

$$T_2 \equiv \omega\pi = 0, \quad T_3 \equiv P\omega = 0, \quad T_4 \equiv P\pi = 0. \quad (61)$$

The Hamiltonian is constructed excluding velocities from the expression

$$H = p_\mu\dot{x} + \pi\dot{\omega} - L + \lambda_i T_i \equiv P\dot{x} + \pi\nabla\omega - L + \lambda_i T_i, \quad (62)$$

where λ_i are the Lagrangian multipliers associated with the primary constraints. From (57) and (58), we observe the equalities $P\dot{x} = (\sqrt{2}L_0)^{-1}(m^2c^2 - \frac{\alpha}{\omega^2})^{\frac{1}{2}}[\dot{x}N\dot{x} - \dot{x}K]$ and $\pi\nabla\omega = (\sqrt{2}L_0)^{-1}(m^2c^2 - \frac{\alpha}{\omega^2})^{\frac{1}{2}}[\nabla\omega N\nabla\omega - \nabla\omega R]$. Together with (59) they imply $P\dot{x} + \pi\nabla\omega = L$. Using this in (62), we conclude that the Hamiltonian is composed from the primary constraints

$$H = \frac{\lambda_1}{2} \left(P^2 + m^2c^2 + \pi^2 - \frac{\alpha}{\omega^2} \right) + \lambda_2(\omega\pi) + \lambda_3(P\omega) + \lambda_4(P\pi). \quad (63)$$

The full set of phase-space coordinates consists of the pairs x^μ, p_μ and ω^μ, π_μ . They fulfill the fundamental Poisson brackets $\{x^\mu, p_\nu\} = \delta^\mu_\nu$ and $\{\omega^\mu, \pi_\nu\} = \delta^\mu_\nu$, then $\{P_\mu, P_\nu\} = R^\sigma_{\lambda\mu\nu}\pi_\sigma\omega^\lambda$, $\{P_\mu, \omega^\nu\} = \Gamma^\nu_{\mu\alpha}\omega^\alpha$, $\{P_\mu, \pi_\nu\} = -\Gamma^\alpha_{\mu\nu}\pi_\alpha$. For the quantities x^μ, P^μ and $S^{\mu\nu}$ these brackets imply the typical relations used by people for spinning particles in Hamiltonian formalism

$$\{x^\mu, P_\nu\} = \delta^\mu_\nu, \quad \{P_\mu, P_\nu\} = -\frac{1}{4}R_{\mu\nu\alpha\beta}S^{\alpha\beta}, \quad \{P_\mu, S^{\alpha\beta}\} = \Gamma^\alpha_{\mu\sigma}S^{\sigma\beta} - \Gamma^\beta_{\mu\sigma}S^{\sigma\alpha}. \quad (64)$$

$$\{S^{\mu\nu}, S^{\alpha\beta}\} = 2(g^{\mu\alpha}S^{\nu\beta} - g^{\mu\beta}S^{\nu\alpha} - g^{\nu\alpha}S^{\mu\beta} + g^{\nu\beta}S^{\mu\alpha}). \quad (65)$$

To reveal the higher-stage constraints and the Lagrangian multipliers we study the equations $\dot{T}_i = \{T_i, H\} = 0$. T_2 implies the secondary constraint

$$\dot{T}_2 = 0 \quad \Rightarrow \quad T_5 \equiv \pi^2 - \frac{\alpha}{\omega^2} \approx 0, \quad (66)$$

then T_1 can be replaced on $P^2 + m^2c^2 \approx 0$. Preservation in time of T_4 and T_3 gives the Lagrangian multipliers λ_3 and λ_4

$$\lambda_3 = 2a\lambda_1(\pi\theta P), \quad \lambda_4 = -2a\lambda_1(\omega\theta P), \quad (67)$$

where we have denoted

$$\theta_{\mu\nu} \equiv R_{\alpha\beta\mu\nu}S^{\alpha\beta}. \quad (68)$$

$$a = \frac{2}{16m^2c^2 + (\theta S)}, \quad (69)$$

Preservation in time of T_1 gives the equation $\lambda_3(\omega\theta P) + \lambda_4(\pi\theta P) = 0$ which is identically satisfied by virtue of (67). No more constraints are generated after this step. We summarize the algebra of Poisson brackets between the constraints in the Table I. T_3 and T_4 represent a pair of second-class constraints, while T_2, T_5 and the combination

$$T_0 = T_1 + 4a(\pi\theta P)T_3 - 4a(\omega\theta P)T_4, \quad (70)$$

TABLE I: Algebra of constraints

	T_1	T_5	T_2	T_3	T_4
$T_1 = P^2 + m^2 c^2$	0	0	0	$\frac{1}{2}(\omega\theta P)$	$\frac{1}{2}(\pi\theta P)$
$T_5 = \pi^2 - \frac{\alpha}{\omega^2}$	0	0	$-2T_5$	$-2T_4$	$-2T_3/\omega^2$
$T_2 = \omega\pi$	0	$2T_5$	0	$-T_3$	T_4
$T_3 = P\omega$	$-\frac{1}{2}(\omega\theta P)$	$2T_4$	T_3	0	$-\frac{1}{8a}$
$T_4 = P\pi$	$-\frac{1}{2}(\pi\theta P)$	$2T_3/\omega^2$	$-T_4$	$\frac{1}{8a}$	0

are the first-class constraints. Taking into account that each second-class constraint rules out one phase-space variable, whereas each first-class constraint rules out two variables, we have the right number of spin degrees of freedom, $8 - (2 + 4) = 2$.

It should be noted that ω^μ and π^μ turns out to be space-like vectors. Indeed, in flat limit and in the frame where $p^\mu = (p^0, \mathbf{0})$ the constraints $\omega p = \pi p = 0$ imply $\omega^0 = \pi^0 = 0$. This implies $\omega^2 \geq 0$ and $\pi^2 \geq 0$. Combining this with the constraint (66), we conclude $\omega^2 > 0$ and $\pi^2 > 0$.

We point out that the first-class constraint $T_5 = \pi^2 - \frac{\alpha}{\omega^2} \approx 0$ can be replaced on the pair

$$\pi^2 = \text{const}, \quad \omega^2 = \text{const}, \quad (71)$$

this gives an equivalent formulation of the model. The Lagrangian which implies the constraints (61) and (71) has been studied in [17, 27, 28]. Hamiltonian and Lagrangian equations for physical variables of the two formulations coincide [29], which proves their equivalence.

Using (67), we can present the Hamiltonian (63) in the form

$$H = \frac{\lambda_1}{2} (P^2 + m^2 c^2 + 4a[(\pi\theta P)(P\omega) - (\omega\theta P)(P\pi)]) + \frac{\lambda_1}{2} \left(\pi^2 - \frac{\alpha}{\omega^2} \right) + \lambda_2(\omega\pi). \quad (72)$$

The dynamics of basic variables is governed by Hamiltonian equations $\dot{z} = \{z, H\}$, where $z = (x, p, \omega, \pi)$, and the Hamiltonian is given in (72). Equivalently, we can use the first-order variational problem equivalent to (55)

$$S_H = \int d\tau \, p_\mu \dot{x}^\mu + \pi_\mu \dot{\omega}^\mu - \left[\frac{\lambda_1}{2} \left(P^2 + (mc)^2 + \pi^2 - \frac{\alpha}{\omega^2} \right) + \lambda_2(\omega\pi) + \lambda_3(P\omega) + \lambda_4(P\pi) \right]. \quad (73)$$

Variation with respect to λ_i gives the constraints (60) and (61), while variation with respect to x, p, ω and π gives the dynamical equations. By construction of S_H , the variational equation $\frac{\delta S_H}{\delta p_\mu} = 0$ is equivalent to $\dot{x}^\mu = \{x^\mu, H\}$, and so on. The equations can be written in a manifestly covariant form as follows:

$$\dot{x}^\mu = \lambda_1 [P^\mu + 2a[(\pi\theta P)\omega^\mu - (\omega\theta P)\pi^\mu]], \quad (74)$$

$$\nabla P_\mu = R^\alpha{}_{\beta\mu\nu} \pi_\alpha \omega^\beta \dot{x}^\nu, \quad (75)$$

$$\nabla \omega^\mu = -2\lambda_1 a (\omega\theta P) P^\mu + \lambda_2 \omega^\mu + \lambda_1 \pi^\mu, \quad \nabla \pi_\mu = -2\lambda_1 a (\pi\theta P) P_\mu - \lambda_2 \pi_\mu - \lambda_1 \frac{\omega_\mu}{\omega^2}. \quad (76)$$

According to general theory [35–37], neither constraints nor equations of motion do not determine the functions λ_1 and λ_2 . Their presence in the equations of motion implies that evolution of our basic variables is ambiguous. This is in correspondence with two local symmetries presented in the model. The variables with ambiguous dynamics do not represent observable quantities, so we need to search for variables that can be candidates for observables. Consider the antisymmetric tensor (49). As a consequence of $T_3 = 0$ and $T_4 = 0$, this obeys the Pirani supplementary condition [4, 5, 7]

$$S^{\mu\nu} P_\nu = 0. \quad (77)$$

Besides, the constraints T_2 and T_5 fix the value of square

$$S^{\mu\nu}S_{\mu\nu} = 8\alpha, \quad (78)$$

so we identify $S^{\mu\nu}$ with the Frenkel spin-tensor [9]. The equations (77) and (78) imply that only two components of spin-tensor are independent, as it should be for spin one-half particle. Equations of motion for $S^{\mu\nu}$ follow from (76). Besides, we express equations (74) and (75) in terms of the spin-tensor. This gives the system

$$\dot{x}^\mu = \lambda_1 [P^\mu + aS^{\mu\beta}\theta_{\beta\alpha}P^\alpha], \quad (79)$$

$$\nabla P_\mu = -\frac{1}{4}R_{\mu\nu\alpha\beta}S^{\alpha\beta}\dot{x}^\nu \equiv -\frac{1}{4}\theta_{\mu\nu}\dot{x}^\nu, \quad (80)$$

$$\nabla S^{\mu\nu} = 2(P^\mu\dot{x}^\nu - P^\nu\dot{x}^\mu), \quad (81)$$

where θ has been defined in (68). Eq. (81), contrary to the equations (76) for ω and π , does not depend on λ_2 . This proves that the spin-tensor is invariant under local spin-plane symmetry. The remaining ambiguity due to λ_1 is related with reparametrization invariance and disappears when we work with physical dynamical variables $x^i(t)$. Equations (79)-(81) together with (77) and (78), form a closed system which determines evolution of a spinning particle.

To obtain the Hamiltonian equations we can equally use the Dirac bracket constructed with help of second-class constraints

$$\{A, B\}_D = \{A, B\} - \frac{1}{8a} [\{A, T_3\}\{T_4, B\} - \{A, T_4\}\{T_3, B\}]. \quad (82)$$

Since the Dirac bracket of a second-class constraint with any quantity vanishes, we can now omit T_3 and T_4 from (72), this yields the Hamiltonian

$$H_1 = \frac{\lambda_1}{2} (P^2 + m^2c^2) + \frac{\lambda_1}{2} \left(\pi^2 - \frac{\alpha}{\omega^2} \right) + \lambda_2(\omega\pi). \quad (83)$$

Then the equations (74)-(76) can be obtained according the rule $\dot{z} = \{z, H_1\}_D$. The quantities x^μ , P^μ and $S^{\mu\nu}$, being invariant under spin-plane symmetry, have vanishing brackets with the corresponding first-class constraints T_2 and T_5 . So, obtaining equations for these quantities, we can omit the last two terms in H_1 , arriving at the familiar relativistic Hamiltonian

$$H_2 = \frac{\lambda_1}{2} (P^2 + m^2c^2). \quad (84)$$

The equations (79)-(81) can be obtained according the rule $\dot{z} = \{z, H_2\}_D$. From (84) we conclude that our model describe spinning particle without gravimagnetic moment. The Hamiltonian with gravimagnetic moment κ have been proposed by Khrilovich [23] adding non minimal interaction $\frac{\lambda_1}{2} \frac{\kappa}{16} R_{\mu\nu\alpha\beta} S^{\mu\nu} S^{\alpha\beta}$ to the expression for H_2 . The corresponding Lagrangian formulation will be constructed in section V A.

Let us exclude momenta P^μ and the auxiliary variable λ_1 from the Hamiltonian equations. This yields second-order equation for the particle's position $x^\mu(\tau)$. To achieve this, we observe that the equation (79) is linear on P

$$\dot{x}^\mu = \lambda_1 T^\mu_\nu P^\nu, \quad \text{with} \quad T^\mu_\nu = \delta^\mu_\nu + aS^{\mu\alpha}\theta_{\alpha\nu}. \quad (85)$$

Using the identity

$$(S\theta S)^{\mu\nu} = -\frac{1}{2}(S\theta)S^{\mu\nu}, \quad \text{where} \quad S\theta = S^{\alpha\beta}\theta_{\alpha\beta}, \quad (86)$$

we find inverse of the matrix T^μ_ν

$$\tilde{T}^\mu_\nu = \delta^\mu_\nu - \frac{1}{8m^2c^2} S^{\mu\sigma}\theta_{\sigma\nu}, \quad T^\mu_\alpha \tilde{T}^\alpha_\nu = \delta^\mu_\nu, \quad (87)$$

so (85) can be solved with respect to P^μ , $P^\mu = \frac{1}{\lambda_1} \tilde{T}^\mu_\nu \dot{x}^\nu$. We substitute P^μ into the constraint $P^2 + m^2c^2 = 0$, this gives expression for λ_1

$$\lambda_1 = \frac{\sqrt{-G_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}{mc} \equiv \frac{\sqrt{-\dot{x}G\dot{x}}}{mc}. \quad (88)$$

We have introduced the effective metric

$$G_{\mu\nu} \equiv \tilde{T}^\alpha_\mu g_{\alpha\beta} \tilde{T}^\beta_\nu. \quad (89)$$

The matrix G is composed from the original metric $\eta_{\mu\nu}$ plus (spin and field-dependent) contribution, $G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(S)$. So we call G the effective metric produced along the world-line by interaction of spin with gravity. The effective metric will play the central role in our discussion of ultra-relativistic limit.

From (85) and (88) we obtain the final expression for P_μ

$$P^\mu = \frac{mc}{\sqrt{-\dot{x}G\dot{x}}} \tilde{T}^\mu_{\nu} \dot{x}^\nu = \frac{mc}{\sqrt{-\dot{x}G\dot{x}}} \left[\dot{x}^\mu - \frac{1}{8m^2c^2} S^{\mu\nu} \theta_{\nu\sigma} \dot{x}^\sigma \right], \quad (90)$$

and Lagrangian form of the Pirani condition

$$S^\mu_{\nu} \dot{x}^\nu - \frac{1}{8(mc)^2} (SS\theta\dot{x})^\mu = 0. \quad (91)$$

Using the equations (90) and (91) in (80) and (81) we finally obtain

$$\nabla \left[\frac{\tilde{T}^\mu_{\nu} \dot{x}^\nu}{\sqrt{-\dot{x}G\dot{x}}} \right] = -\frac{1}{4mc} R^\mu_{\nu\alpha\beta} S^{\alpha\beta} \dot{x}^\nu, \quad (92)$$

$$\nabla S^{\mu\nu} = \frac{1}{4mc\sqrt{-\dot{x}G\dot{x}}} \dot{x}^{[\mu} S^{\nu]\sigma} \theta_{\sigma\alpha} \dot{x}^\alpha. \quad (93)$$

These equations, together with the conditions (91) and (78), form closed system for the set $(x^\mu, S^{\mu\nu})$. The consistency of the constraints (91) and (78) with the dynamical equations is guaranteed by Dirac procedure for singular systems.

B. Lagrangian action of spinning particle with unfixed value of spin

The Lagrangians (47) and (55) yield the fixed value of spin (78), that is they correspond to an elementary particle. Let us present the modification which leads to the theory with unfixed spin, and, similarly to Hanson-Regge approach [20], with a mass-spin trajectory constraint. Consider the following Lagrangian in curved background

$$L = -\frac{mc}{\sqrt{2}} \sqrt{-\dot{x}N\dot{x} - l^2 \frac{\nabla\omega N \nabla\omega}{\omega^2}} + \sqrt{\left[\dot{x}N\dot{x} + l^2 \frac{\nabla\omega N \dot{\omega}}{\omega^2} \right]^2 - 4l^2 \frac{(\dot{x}N \nabla\omega)^2}{\omega^2}}, \quad (94)$$

where l is a parameter with the dimension of length. Applying the Dirac procedure as in Section IV A, we obtain the Hamiltonian

$$H = \frac{\lambda_1}{2} \left(P^2 + m^2c^2 + \frac{\pi^2\omega^2}{l^2} \right) + \lambda_2(\omega\pi) + \lambda_3(P\omega) + \lambda_4(P\pi), \quad (95)$$

which turns out to be combination of the first-class constraints $P^2 + m^2c^2 + \frac{\pi^2\omega^2}{l^2} = 0$, $\omega\pi = 0$ and the second-class constraints $P\omega = 0$, $P\pi = 0$. The Dirac procedure stops on the first stage, that is there are no of secondary constraints. As compared with (55), the first-class constraint $\pi^2 - \frac{\alpha}{\omega^2} = 0$ does not appear in the present model. Due to this, square of spin is not fixed, $S^2 = 8(\omega^2\pi^2 - \omega\pi) \approx 8\omega^2\pi^2$. Using this equality, the mass-shell constraint acquires the string-like form

$$P^2 + m^2c^2 + \frac{1}{8l^2} S^2 = 0. \quad (96)$$

The model has four physical degrees of freedom in the spin-sector. As the independent gauge-invariant degrees of freedom, we can take three components S^{ij} of the spin-tensor together with any one product of conjugate coordinates, for instance, $\omega^0\pi^0$.

Using the auxiliary variable λ , we can rewrite the Lagrangian in the equivalent form

$$L = \frac{1}{2\lambda} \left[\dot{x}N\dot{x} + l^2 \frac{\nabla\omega N \nabla\omega}{\omega^2} - \sqrt{\left[\dot{x}N\dot{x} + l^2 \frac{\nabla\omega N \dot{\omega}}{\omega^2} \right]^2 - 4l^2 \frac{(\dot{x}N \nabla\omega)^2}{\omega^2}} \right] - \frac{\lambda}{4} m^2c^2. \quad (97)$$

Contrary to Eq. (94), it admits the massless limit.

C. Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations and dynamics of representative point of a rotating body

In this section we discuss MPTD-equations of a rotating body in the form studied by Dixon⁷ (for the relation of the Dixon equations with those of Papapetrou and Tulczyjew see p. 335 in [5])

$$\nabla P^\mu = -\frac{1}{4}R^\mu{}_{\nu\alpha\beta}S^{\alpha\beta}\dot{x}^\nu \equiv -\frac{1}{4}(\theta\dot{x})^\mu, \quad (98)$$

$$\nabla S^{\mu\nu} = 2(P^\mu\dot{x}^\nu - P^\nu\dot{x}^\mu), \quad (99)$$

$$S^{\mu\nu}P_\nu = 0, \quad (100)$$

and compare them with equations of motion of our spinning particle. In particular, we show that the effective metric $G_{\mu\nu}$ also emerges in this formalism. MPTD-equations appeared in multipole approach to description of a body [1–6], where the energy-momentum of the body is modelled by a set of multipoles. In this approach $x^\mu(\tau)$ is called representative point of the body, we take it in arbitrary parametrization τ (contrary to Dixon, we do not assume the proper-time parametrization, that is we do not add the equation $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -c^2$ to the system above). $S^{\mu\nu}(\tau)$ is associated with inner angular momentum, and $P^\mu(\tau)$ is called momentum. The first-order equations (98) and (99) appear in the pole-dipole approximation, while the algebraic equation (100) has been added by hand⁸. After that, the number of equations coincides with the number of variables.

To compare MPTD-equations with those of section IV A, we first observe some useful consequences of the system (98)-(100).

Take derivative of the constraint, $\nabla(S^{\mu\nu}P_\nu) = 0$, and use (98) and (99), this gives the expression

$$(P\dot{x})P^\mu = P^2\dot{x}^\mu + \frac{1}{8}(S\theta\dot{x})^\mu, \quad (101)$$

which can be written in the form

$$P^\mu = \frac{P^2}{(P\dot{x})} \left(\delta^\mu{}_\nu + \frac{1}{8P^2}(S\theta)^\mu{}_\nu \right) \dot{x}^\nu \equiv \frac{P^2}{(P\dot{x})} \tilde{T}^\mu{}_\nu \dot{x}^\nu. \quad (102)$$

Contract (101) with \dot{x}_μ . Taking into account that $(P\dot{x}) < 0$, this gives $(P\dot{x}) = -\sqrt{-P^2}\sqrt{-\dot{x}\tilde{T}\dot{x}}$. Using this in Eq. (102) we obtain

$$P^\mu = \frac{\sqrt{-P^2}}{\sqrt{-\dot{x}\tilde{T}\dot{x}}}(\tilde{T}\dot{x})^\mu, \quad \tilde{T}^\mu{}_\nu = \delta^\mu{}_\nu + \frac{1}{8P^2}(S\theta)^\mu{}_\nu. \quad (103)$$

For the latter use we observe that in our model with composite $S^{\mu\nu}$ we used the identity (86) to invert $T^\mu{}_\nu$, then the Hamiltonian equation (79) has been written in the form (90), the latter can be compared with (103).

Contracting (99) with $S_{\mu\nu}$ and using (100) we obtain $\frac{d}{d\tau}(S^{\mu\nu}S_{\mu\nu}) = 0$, that is, square of spin is a constant of motion. Contraction of (101) with P_μ gives $(PS\theta\dot{x}) = 0$. Contraction of (101) with $(\dot{x}\theta)_\mu$ gives $(P\theta\dot{x}) = 0$. Contraction of (98) with P_μ , gives $\frac{d}{d\tau}(P^2) = -\frac{1}{2}(P\theta\dot{x}) = 0$, that is P^2 is one more constant of motion, say k , $\sqrt{-P^2} = k = \text{const}$ (in our model this is fixed as $k = mc$). Substituting (103) into the equations (98)-(100) we now can exclude P^μ from these equations, modulo to the constant of motion $k = \sqrt{-P^2}$.

Thus, square of momentum can not be excluded from the system (98)-(101), that is MPTD-equations in this form do not represent a Hamiltonian system for the pair x^μ, P^μ . To improve this point, we note that Eq. (103) acquires a conventional form (as the expression for conjugate momenta of x^μ in the Hamiltonian formalism), if we add to the system (98)-(100) one more equation, which fixes the remaining quantity P^2 (Dixon noticed this for the body in electromagnetic field, see his eq. (4.5) in [6]). To see, how the equation could look, we note that for non-rotating body (pole approximation) we expect equations of motion of spinless particle, $\nabla p^\mu = 0$, $p^\mu = \frac{mc}{\sqrt{-\dot{x}g\dot{x}}}\dot{x}^\mu$, $p^2 + (mc)^2 = 0$. Independent equations of the system (98)-(101) in this limit read $\nabla P^\mu = 0$, $P^\mu = \frac{\sqrt{-P^2}}{\sqrt{-\dot{x}g\dot{x}}}\dot{x}^\mu$. Comparing the two systems, we see that the missing equation is the mass-shell condition $P^2 + (mc)^2 = 0$. Returning to the pole-dipole

⁷ Our S is twice of that of Dixon.

⁸ For geometric interpretation of the spin supplementary condition in the multipole approach see [5].

approximation, an admissible equation should be $P^2 + (mc)^2 + f(S, \dots) = 0$, where f must be a constant of motion. Since the only constant of motion in arbitrary background is S^2 , we have finally

$$P^2 = -(mc)^2 - f(S^2). \quad (104)$$

With this value of P^2 , we can exclude P^μ from MPTD-equations, obtaining closed system with second-order equation for x^μ (so we refer the resulting equations as Lagrangian form of MPTD-equations). We substitute (103) into (98)-(100), this gives

$$\nabla \frac{(\tilde{\mathcal{T}}\dot{x})^\mu}{\sqrt{-\dot{x}\tilde{\mathcal{T}}\dot{x}}} = -\frac{1}{4\sqrt{-P^2}}(\theta\dot{x})^\mu, \quad (105)$$

$$\nabla S^{\mu\nu} = -\frac{1}{4\sqrt{-P^2}\sqrt{-\dot{x}\tilde{\mathcal{T}}\dot{x}}}\dot{x}^{[\mu}(S\theta\dot{x})^{\nu]}, \quad (106)$$

$$(SS\theta\dot{x})^\mu = -8P^2(S\dot{x})^\mu, \quad (107)$$

where (104) is implied. They determine evolution of x^μ and $S^{\mu\nu}$ for each given function $f(S^2)$.

It is convenient to introduce the effective metric \mathcal{G} composed from the "tetrad field" $\tilde{\mathcal{T}}^\mu_\nu$

$$\mathcal{G}_{\mu\nu} \equiv g_{\alpha\beta}\tilde{\mathcal{T}}^\alpha_\mu\tilde{\mathcal{T}}^\beta_\nu. \quad (108)$$

Eq. (107) implies the identity

$$\dot{x}\tilde{\mathcal{T}}\dot{x} = \dot{x}\mathcal{G}\dot{x}, \quad (109)$$

so we can replace $\sqrt{-\dot{x}\tilde{\mathcal{T}}\dot{x}}$ in (105)-(107) by $\sqrt{-\dot{x}\mathcal{G}\dot{x}}$.

In resume, we have presented MPTD-equations in the form

$$P^\mu = \frac{\sqrt{-P^2}}{\sqrt{-\dot{x}\mathcal{G}\dot{x}}}(\tilde{\mathcal{T}}\dot{x})^\mu, \quad \nabla P^\mu = -\frac{1}{4}(\theta\dot{x})^\mu, \quad \nabla S^{\mu\nu} = 2P^{[\mu}\dot{x}^{\nu]}, \quad S^{\mu\nu}P_\nu = 0, \quad (110)$$

$$P^2 + (mc)^2 + f(S^2) = 0, \quad (111)$$

$$S^2 \text{ is a constant of motion,} \quad (112)$$

with $\tilde{\mathcal{T}}^\mu_\nu$ given in (103). Now we are ready to compare them with Hamiltonian equations of our spinning particle, which we write here in the form

$$P^\mu = \frac{mc}{\sqrt{-\dot{x}G\dot{x}}}(\tilde{T}\dot{x})^\mu, \quad \nabla P^\mu = -\frac{1}{4}(\theta\dot{x})^\mu, \quad \nabla S^{\mu\nu} = 2P^{[\mu}\dot{x}^{\nu]}, \quad S^{\mu\nu}P_\nu = 0, \quad (113)$$

$$P^2 + (mc)^2 = 0, \quad (114)$$

$$S^2 = 8\alpha, \quad (115)$$

with \tilde{T}^μ_ν given in (87). Comparing the systems, we see that our spinning particle has fixed values of spin and canonical momentum, while for MPTD-particle the spin is a constant of motion and momentum is a function of spin. We conclude that all the trajectories of a body with given m and $S^2 = \beta$ are described by our spinning particle with spin $\alpha = \frac{\beta}{8}$ and with the mass equal to $\sqrt{m^2 - \frac{f^2(\beta)}{c^2}}$. In this sense our spinning particle is equivalent to MPTD-particle.

We point out that our final conclusion remains true even if we do not add (104) to MPTD-equations: to study the class of trajectories of a body with $\sqrt{-P^2} = k$ and $S^2 = \beta$ we take our spinning particle with $m = \frac{k}{c}$ and $\alpha = \frac{\beta}{8}$.

MPTD-equations in the Lagrangian form (105)-(107) can be compared with (91)-(93).

D. Ultra-relativistic limit: the problems with MPTD-equations

The equations for trajectory (92) and for precession of spin (93) became singular at critical velocity which obeys the equation

$$\dot{x}G\dot{x} = 0. \quad (116)$$

As we discussed in Introduction, the singularity determines behavior of the particle in ultra-relativistic limit. In Eq. (116) appeared the effective metric (89) instead of the original metric $g_{\mu\nu}$. It should be noted that the incorporation of the constraints (61) and (66) into a variational problem, as well as the search for an interaction consistent with them represent very strong restrictions on possible form of the Lagrangian. So, the appearance of effective metric seems to be unavoidable in a systematically constructed model of spinning particle. The same conclusion follow from our analysis of MPTD-equations in section IV C.

The effective metric is composed from the original one plus (spin and field-dependent) contribution, $G = g + h(S)$. So we need to decide, which of them the particle probes as the space-time metric. Let us consider separately the two possibilities.

Let us use g to define the three-dimensional geometry (5)-(8). This leads to two problems. The first problem is that the critical speed turns out to be slightly more then the speed of light. To see this, we use the Pirani condition to write (116) in the form

$$-\left(\frac{dt}{dx^0}\right)^2 \dot{x}G\dot{x} = (c^2 - \mathbf{v}\gamma\mathbf{v}) + \frac{1}{(2m^2c^2)^2} (v\theta SS\theta v) = 0, \quad (117)$$

with v^μ defined in (10). Using the expression $S^{\mu\nu} = 2\omega^{[\mu}\pi^{\nu]}$, we obtain

$$-\left(\frac{dt}{dx^0}\right)^2 \dot{x}G\dot{x} = (c^2 - \mathbf{v}\gamma\mathbf{v}) + \frac{1}{(m^2c^2)^2} (\pi^2(v\theta\omega)^2 + \omega^2(v\theta\pi)^2) = 0. \quad (118)$$

As π and ω are space-like vectors (see the discussion below Eq. (70)), the last term is non-negative, this implies $|\mathbf{v}_{cr}| \geq c$. Let us confirm that generally this term is nonvanishing function of velocity, then $|\mathbf{v}_{cr}| > c$. Assume the contrary, that this term vanishes at some velocity, then

$$v\theta\omega = \theta_{0i}\omega^i + \theta_{i0}v^i\omega^0 = 0, \quad (119)$$

$$v\theta\pi = \theta_{0i}\pi^i + \theta_{i0}v^i\pi^0 = 0. \quad (120)$$

We analyze these equations in the following special case. Consider a space with covariantly-constant curvature $\nabla_\mu R_{\mu\nu\alpha\beta} = 0$. Then $\frac{d}{d\tau}(\theta_{\mu\nu}S^{\mu\nu}) = 2\theta_{\mu\nu}\nabla S^{\mu\nu}$, and using (93) we conclude that $\theta_{\mu\nu}S^{\mu\nu}$ is an integral of motion. We further assume that the only non vanishing part is the electric [38] part of the curvature, $R_{0i0j} = K_{ij}$, with $\det K_{ij} \neq 0$. Then the integral of motion acquires the form

$$\theta_{\mu\nu}S^{\mu\nu} = 2K_{ij}S^{0i}S^{0j}. \quad (121)$$

Let us take the initial conditions for spin such that $K_{ij}S^{0i}S^{0j} \neq 0$, then this holds at any future instant. Contrary to this, the system (119) implies $K_{ij}S^{0i}S^{0j} = 0$. Thus, the critical speed does not always coincide with the speed of light and, in general case, we expect that \mathbf{v}_{cr} is both field and spin-dependent quantity.

The second problem is that acceleration of MPTD-particle grows up in the ultra-relativistic limit. In the spinless limit the equation (92) turn into the geodesic equation. Spin causes deviations from the geodesic equation due to right hand side of this equation, as well as due to the presence of the tetrad field $\tilde{T}^\mu{}_\nu$ and of the effective metric G in the left hand side. Due to the dependence of the tetrad field on the spin-tensor S , the singularity presented in (93) causes the appearance of the term proportional to $\frac{1}{\sqrt{\dot{x}G\dot{x}}}$ in the expression for longitudinal acceleration. In the result, the acceleration grows up to infinity as the particle's speed approximates to the critical speed. To see this, we separate derivative of $\tilde{T}^\mu{}_\nu$ in Eq. (92)

$$\nabla \left[\frac{\dot{x}^\mu}{\sqrt{-\dot{x}G\dot{x}}} \right] = -T^\mu{}_\alpha \left(\nabla \tilde{T}^\alpha{}_\beta \right) \frac{\dot{x}^\beta}{\sqrt{-\dot{x}G\dot{x}}} - \frac{1}{4mc} T^\mu{}_\nu (\theta \dot{x})^\nu. \quad (122)$$

Using (93) we obtain

$$\left[\nabla \tilde{T}^\mu{}_\nu \right] \dot{x}^\nu = -\frac{S^{\mu\alpha}}{8m^2c^2} \left[\frac{R_{\alpha\nu\beta\sigma} \dot{x}^\beta (S\theta \dot{x})^\sigma}{2mc\sqrt{-\dot{x}G\dot{x}}} + S^{\beta\sigma} (\nabla R_{\alpha\nu\beta\sigma}) \right] \dot{x}^\nu. \quad (123)$$

Using this expression together with the identity $(TS)^{\mu\nu} = 8m^2c^2aS^{\mu\nu}$, the equation (122) reads

$$\frac{d}{d\tau} \left[\frac{\dot{x}^\mu}{\sqrt{-\dot{x}G\dot{x}}} \right] = \frac{f^\mu}{\sqrt{-\dot{x}G\dot{x}}}, \quad (124)$$

where we denoted

$$f^\mu \equiv aS^{\mu\alpha} \left[\frac{R_{\alpha\nu\beta\sigma}\dot{x}^\beta(S\theta\dot{x})^\sigma}{2mc\sqrt{-\dot{x}G\dot{x}}} + S^{\beta\sigma}(\nabla R_{\alpha\nu\beta\sigma}) \right] \dot{x}^\nu - (\Gamma\dot{x})^\mu - \frac{\sqrt{-\dot{x}G\dot{x}}}{4mc} (T\theta\dot{x})^\mu. \quad (125)$$

It will be sufficient to consider static metric $g_{\mu\nu}(\mathbf{x})$ with $g_{0i} = 0$. Then three-dimensional metric and velocity are

$$\gamma_{ij} = g_{ij}, \quad v^i = \frac{c}{\sqrt{-g_{00}}} \frac{dx^i}{dx^0}, \quad (126)$$

Taking $\tau = x^0$, the spacial part of equation (124) with this metric reads

$$\left(\frac{dt}{dx^0} \right)^{-1} \frac{d}{dx^0} \left[\frac{v^i}{\sqrt{-vGv}} \right] = \frac{f^i(v)}{\sqrt{-vGv}}. \quad (127)$$

with v^μ defined in (10), for the case

$$v^\mu = \left(\frac{c}{\sqrt{-g_{00}}}, \mathbf{v} \right), \quad (128)$$

and

$$-vGv = -v\tilde{T}v = c^2 - \mathbf{v}g\mathbf{v} + \frac{(vS\theta v)}{8m^2c^2}. \quad (129)$$

In the result, we have presented the equation for trajectory in the form convenient for analysis of acceleration, see (28). Using the definition of three-dimensional covariant derivative (25), we present the derivative on the l.h.s. of (127) as follows

$$\frac{d}{dx^0} \left[\frac{v^i}{\sqrt{-vGv}} \right] = \frac{1}{\sqrt{-vGv}} \left[\mathcal{M}^i_k \nabla_0 v^k - \tilde{\Gamma}(\gamma)^i_{jk} v^j v^k \frac{dt}{dx^0} + \frac{Kv^i}{2(-vGv)} \right], \quad (130)$$

We have denoted

$$K = (\nabla_0 G_{\mu\nu})v^\mu v^\nu - v^\mu G_{\mu 0} v^k \partial_k \ln(-g_{00}), \quad \mathcal{M}^i_k = \delta^i_k - \frac{v^i v^\mu G_{\mu k}}{vGv}. \quad (131)$$

The matrix \mathcal{M}^i_k has the inverse

$$\tilde{\mathcal{M}}^i_k = \delta^i_k + \frac{v^i v^\mu G_{\mu k}}{v^\sigma G_{\sigma 0} v^0}, \quad \text{then} \quad \tilde{\mathcal{M}}^i_k v^k = v^i \frac{vGv}{v^\sigma G_{\sigma 0} v^0}. \quad (132)$$

Combining these equations, we obtain the three-acceleration of our spinning particle

$$a^i = \left(\frac{dt}{dx^0} \right)^{-1} \nabla_0 v^i = \tilde{\mathcal{M}}^i_k \left[f^k + (\tilde{\Gamma}v v)^k \right] + \frac{Kv^i}{2v^\sigma G_{\sigma 0}}. \quad (133)$$

Finally, using manifest form of f^i from (125) we have

$$a^i = \frac{\tilde{\mathcal{M}}^i_k \hat{S}^k}{\sqrt{-vGv}} - c^2 \tilde{\mathcal{M}}^i_k \frac{\gamma^{kj} \partial_j g_{00}}{2g_{00}} - \frac{\sqrt{-vGv}}{4mc} \tilde{\mathcal{M}}^i_k (T\theta v)^k + \frac{Kv^i}{2v^\sigma G_{\sigma 0}} + a \tilde{\mathcal{M}}^i_k S^{k\alpha} R_{\alpha\nu\beta\sigma;\lambda} S^{\beta\sigma} v^\nu v^\lambda. \quad (134)$$

The longitudinal acceleration is obtained by projecting a^i on the direction of velocity, that is

$$\begin{aligned} (\mathbf{v}\gamma\mathbf{a}) = & \frac{ac(\mathbf{v}\gamma\tilde{\mathcal{M}})_k \hat{S}^k}{2m\sqrt{-vGv}} - c^2 (\mathbf{v}\gamma\tilde{\mathcal{M}})_k \frac{\gamma^{kj} \partial_j g_{00}}{2g_{00}} - \frac{\sqrt{-vGv}}{4mc} (\mathbf{v}\gamma\tilde{\mathcal{M}})_k (T\theta v)^k + \\ & \frac{K}{2v^\sigma G_{\sigma 0}} (\mathbf{v}\gamma\mathbf{v}) + a (\mathbf{v}\gamma\tilde{\mathcal{M}})_k S^{k\alpha} R_{\alpha\nu\beta\sigma;\lambda} S^{\beta\sigma} v^\nu v^\lambda. \end{aligned} \quad (135)$$

where $\hat{S}^k = S^{k\mu} R_{\mu\nu\alpha\beta} v^\nu v^\alpha (S\theta v)^\beta$. As the speed of the particle closes to the critical velocity, the longitudinal acceleration diverges due to the first term in (135). In resume, assuming that MPTD-particle sees the original geometry $g_{\mu\nu}$, we have a theory with unsatisfactory behavior in the ultra-relativistic limit.

Let us consider the second possibility, that is we take $G_{\mu\nu}$ to construct the three-dimensional geometry (5)-(8). With these definitions we have, by construction, $-\dot{x}G\dot{x} = (\frac{dt}{dx^0})^2(c^2 - (\mathbf{v}\gamma\mathbf{v}))$, so the critical speed coincides with the speed of light. In the present case, the expression for three-acceleration can be obtained in closed form for an arbitrary curved background. Taking $\tau = x^0$ the spacial part of (124) implies

$$\left(\frac{dt}{dx^0}\right)^{-1} \frac{d}{dx^0} \left[\frac{v^i}{\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}} \right] = \frac{f^i(v)}{\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}}. \quad (136)$$

where, from (125), f^i is given by

$$f^i \equiv aS^{i\alpha} \left[\frac{R_{\alpha\nu\beta\sigma} v^\beta (S\theta v)^\sigma}{2mc\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}} + S^{\beta\sigma} (\nabla R_{\alpha\nu\beta\sigma}) \right] v^\nu - \Gamma^i_{\mu\nu}(G) v^\mu v^\nu - \frac{\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}}{4mc} (T\theta v)^i. \quad (137)$$

Equation (136) is of the form (28), so the acceleration is given by (37) and (38) where, for the present case, $\gamma_{ij} = G_{ij} - \frac{G_{0i}G_{0j}}{G_{00}}$

$$a^i = \tilde{M}^i_j [f^j + \tilde{\Gamma}^j_{kl}(\gamma) v^k v^l] + \frac{1}{2} \left(\frac{dt}{dx^0} \right)^{-1} \left[(\mathbf{v}\partial_0\gamma\gamma^{-1})^i - \frac{(\mathbf{v}\partial_0\gamma\mathbf{v})}{c^2} v^i \right], \quad (138)$$

$$\mathbf{v}\gamma\mathbf{a} = \left(1 - \frac{\mathbf{v}\gamma\mathbf{v}}{c^2} \right) \left[(\mathbf{v}\gamma)_i [f^i(v) + \tilde{\Gamma}^i_{kl}(\gamma) v^k v^l] + \frac{1}{2} \left(\frac{dt}{dx^0} \right)^{-1} (\mathbf{v}\partial_0\gamma\mathbf{v}) \right]. \quad (139)$$

With f^i given in (137), the longitudinal acceleration vanishes as $v \rightarrow c$.

Let us resume the results of this subsection. Assuming that spinning particle probes the three-dimensional space-time geometry determined by the original metric g , we have a theory with unsatisfactory ultra-relativistic limit. First, the critical speed, which the particle can not overcome during its evolution in gravitational field, can be more then the speed of light. Second, the longitudinal acceleration grows up to infinity in the ultra-relativistic limit. Assuming that the the particle sees the effective metric $G(S)$ as the space-time metric, we avoided the two problems. But the resulting theory still possess the problem. The acceleration (138) contains the singularity due to $f^i \sim \frac{1}{\sqrt{c^2 - (\mathbf{v}\gamma\mathbf{v})}}$, that is at $v = c$ the acceleration becomes orthogonal to the velocity, but remains divergent. We conclude that MPTD-equations do not seem promising candidate for description of a relativistic rotating body.

V. NON MINIMAL INTERACTION WITH GRAVITATIONAL FIELD.

Can we modify the MPTD-equations to obtain a theory with reasonable behavior with respect to the original metric $g_{\mu\nu}$? In the previous section we have noticed that the bad behavior of acceleration originates from the fact that variation rate of spin (93) diverges in the ultra-relativistic limit, $\nabla S \sim \frac{1}{\sqrt{\dot{x}G\dot{x}}}$, and contributes into the expression for acceleration (135) through the tetrad field $\tilde{T}^\mu_\nu(S)$. To improve this, we remind that MPTD-equations result from minimal interaction of spinning particle with gravitational field. In this section we demonstrate that vector model of spin admits also a non-minimal interaction which involves the interaction constant κ . By analogy with the magnetic moment, the interaction constant κ is called gravimagnetic moment [23]. In the resulting theory, the equation for precession of spin, $\nabla S \sim \frac{1}{\sqrt{-\dot{x}G\dot{x}}}$, is replaced by $\nabla S \sim \sqrt{-\dot{x}g\dot{x}}$. This improves the bad behavior of MPTD equations. As it will be seen below, introducing κ we effectively change the supplementary spin condition and hence the definition of center of mass.

A. Hamiltonian variational problem

We add the term $\frac{\lambda_1}{2} \kappa R_{\alpha\beta\mu\nu} \omega^\alpha \pi^\beta \omega^\mu \pi^\nu \equiv \frac{\lambda_1}{2} \frac{\kappa}{16} \theta S$ into the Hamiltonian action (73). Thus we consider the variational problem

$$S_\kappa = \int d\tau \left(p_\mu \dot{x}^\mu + \pi_\mu \dot{\omega}^\mu - \left[\frac{\lambda_1}{2} \left(P^2 + \kappa R_{\alpha\beta\mu\nu} \omega^\alpha \pi^\beta \omega^\mu \pi^\nu + (mc)^2 + \pi^2 - \frac{\alpha}{\omega^2} \right) + \lambda_2(\omega\pi) + \lambda_3(P\omega) + \lambda_4(P\pi) \right] \right) \quad (140)$$

on the space of independent variables $x^\mu, p_\nu, \omega^\mu, \pi_\nu$ and λ_a . We have denoted $P_\mu \equiv p_\mu - \Gamma_{\alpha\mu}^\beta \omega^\alpha \pi_\beta$, $P^2 = g^{\mu\nu} P_\mu P_\nu$ and so on. Note also that the first two terms can be identically rewritten in the general-covariant form $p_\mu \dot{x}^\mu + \pi_\mu \dot{\omega}^\mu = P_\mu \dot{x}^\mu + \pi_\mu \nabla \omega^\mu$. Variation of the action with respect to λ_a gives the algebraic equations

$$P^2 + \kappa R_{\alpha\beta\mu\nu} \omega^\alpha \pi^\beta \omega^\mu \pi^\nu + (mc)^2 + \pi^2 - \frac{\alpha}{\omega^2} = 0, \quad (141)$$

$$\omega\pi = 0, \quad P\omega = 0, \quad P\pi = 0, \quad (142)$$

while variations with respect to the remaining variables yield dynamical equations which can be written in the covariant form as follows⁹

$$\frac{\delta S_\kappa}{\delta p_\mu} = 0 \quad \Leftrightarrow \quad \dot{x}^\mu = \lambda_1 P^\mu + \lambda_3 \omega^\mu + \lambda_4 \pi^\mu, \quad (143)$$

$$\frac{\delta S_\kappa}{\delta x^\mu} = 0 \quad \Leftrightarrow \quad \nabla P_\mu = -R_{\mu\nu\alpha\beta} \dot{x}^\nu \omega^\alpha \pi^\beta - \frac{1}{2} \lambda_1 \kappa \nabla_\mu R_{\sigma\nu\alpha\beta} \omega^\sigma \pi^\nu \omega^\alpha \pi^\beta, \quad (144)$$

$$\frac{\delta S_\kappa}{\delta \pi_\mu} = 0 \quad \Leftrightarrow \quad \nabla \omega^\mu = \lambda_1 \pi^\mu - \lambda_1 \kappa R^\mu{}_{\alpha\beta\nu} \omega^\alpha \omega^\beta \pi^\nu + \lambda_2 \omega^\mu + \lambda_4 P^\mu, \quad (145)$$

$$\frac{\delta S_\kappa}{\delta \omega^\mu} = 0 \quad \Leftrightarrow \quad \nabla \pi_\mu = -\frac{\lambda_1 \alpha}{\omega^4} \omega_\mu - \lambda_1 \kappa R_{\mu\nu\alpha\beta} \pi^\nu \omega^\alpha \pi^\beta - \lambda_2 \pi_\mu - \lambda_3 P_\mu. \quad (146)$$

Eq. (143) has been repeatedly used to obtain the final form (144)-(146) of the equations $\frac{\delta S_\kappa}{\delta x^\mu} = 0$, $\frac{\delta S_\kappa}{\delta \pi_\mu} = 0$ and $\frac{\delta S_\kappa}{\delta \omega^\mu} = 0$. Computing time-derivative of the algebraic equations (142) and using (143)-(146) we obtain the consequences

$$\pi^2 - \frac{\alpha}{\omega^2} = 0, \quad (147)$$

$$\begin{aligned} \lambda_3 &= 4a\lambda_1 [2(1-\kappa)R_{\alpha\beta\mu\nu} \omega^\alpha \pi^\beta \pi^\mu P^\nu + \kappa \pi^\sigma (\nabla_\sigma R_{\mu\nu\alpha\beta}) \omega^\mu \pi^\nu \omega^\alpha \pi^\beta], \\ \lambda_4 &= -4a\lambda_1 [2(1-\kappa)R_{\alpha\beta\mu\nu} \omega^\alpha \pi^\beta \omega^\mu P^\nu + \kappa \omega^\sigma (\nabla_\sigma R_{\mu\nu\alpha\beta}) \omega^\mu \pi^\nu \omega^\alpha \pi^\beta], \end{aligned} \quad (148)$$

Here and below we use the following notation. In the equation which relates velocity and momentum will appear the matrix \mathcal{T}^μ_ν

$$\mathcal{T}^\alpha{}_\nu \equiv \delta^\alpha{}_\nu - (\kappa - 1)aS^{\alpha\sigma}\theta_{\sigma\nu}, \quad a = \frac{2}{16m^2c^2 + (\kappa + 1)(S\theta)}. \quad (149)$$

The matrix has an inverse given by

$$\tilde{\mathcal{T}}^\alpha{}_\nu \equiv \delta^\alpha{}_\nu + (\kappa - 1)bS^{\alpha\sigma}\theta_{\sigma\nu}, \quad b = \frac{1}{8m^2c^2 + \kappa(S\theta)}. \quad (150)$$

The vector Z^μ is defined by

$$Z^\mu = \frac{b}{8c} S^{\mu\sigma} (\nabla_\sigma R_{\alpha\beta\rho\delta}) S^{\alpha\beta} S^{\rho\delta} \equiv \frac{b}{8c} S^{\mu\sigma} \nabla_\sigma (S\theta). \quad (151)$$

This vanishes in a space with homogeneous curvature, $\nabla R = 0$.

The time-derivatives of (141), (147) and (148) do not yield new algebraic equations. Due to (147) we can replace the constraint (141) on $P^2 + \kappa R_{\alpha\beta\mu\nu} \omega^\alpha \pi^\beta \omega^\mu \pi^\nu + (mc)^2 = 0$. The obtained expressions for λ_3 and λ_4 can be used to exclude these variables from the equations (143)-(146).

⁹ Note that by construction of S_κ , the variational equation $\frac{\delta S_\kappa}{\delta p_\mu} = 0$ is equivalent to $\dot{x}^\mu = \{x^\mu, H\}$, and so on.

The constraints and equations of motion do not determine the functions λ_1 and λ_2 , that is the non-minimal interaction preserves both reparametrization and spin-plane symmetries of the theory. The presence of λ_1 and λ_2 in the equations (145) and (146) implies that evolution of the basic variables is ambiguous, so they are not observable. To find the candidates for observables, we note once again that (145) and (146) imply an equation for $S^{\mu\nu}$ which does not contain λ_2 . So we rewrite (143) and (144) in terms of spin-tensor and add to them the equation for $S^{\mu\nu}$, this gives the system

$$\dot{x}^\mu = \lambda_1 \left[\mathcal{T}^\mu{}_\nu P^\nu + \kappa \frac{ac}{b} Z^\mu \right], \quad (152)$$

$$\nabla P_\mu = -\frac{1}{4} \theta_{\mu\nu} \dot{x}^\nu - \frac{\lambda_1 \kappa}{32} \nabla_\mu (S\theta), \quad (153)$$

$$\nabla S^{\mu\nu} = -\frac{\kappa \lambda_1}{4} (\theta S)^{[\mu\nu]} + 2P^{[\mu} \dot{x}^{\nu]}. \quad (154)$$

Besides, the constraints (141), (142) and (147) imply

$$P^2 + \frac{\kappa}{16} \theta S + (mc)^2 = 0, \quad (155)$$

$$S^{\mu\nu} P_\nu = 0, \quad S^{\mu\nu} S_{\mu\nu} = 8\alpha. \quad (156)$$

The equations (156) imply that only two components of spin-tensor are independent, as it should be for spin one-half particle. Eq. (154), contrary to the equations for ω and π , does not depend on λ_2 . This proves that the spin-tensor is invariant under local spin-plane symmetry. The remaining ambiguity due to λ_1 is related with reparametrization invariance and disappears when we work with physical dynamical variables $x^i(t)$. Equations (152)-(154), together with (155) and (156), form a closed system which determines evolution of the spinning particle.

The gravimagnetic moment κ can generally take any value. When $\kappa = 0$ we recover the MPTD equations.

Let us exclude P^μ and λ_1 from the equations (153) and (154). Using (150) we solve (152) with respect to P^μ . Using the resulting expression in the constraint (155) we obtain λ_1

$$\lambda_1 = \frac{\sqrt{-\dot{x}\mathcal{G}\dot{x}}}{m_r c}, \quad \text{with} \quad m_r^2 \equiv m^2 + \frac{\kappa}{16c^2} (S\theta) - \kappa^2 Z^2, \quad (157)$$

where the effective metric now is given by

$$\mathcal{G}_{\mu\nu} = \tilde{\mathcal{T}}^\alpha{}_\mu g_{\alpha\beta} \tilde{\mathcal{T}}^\beta{}_\nu. \quad (158)$$

Then the expression for momentum in terms of velocity implied by (152) is

$$P^\mu = \frac{m_r c}{\sqrt{-\dot{x}\mathcal{G}\dot{x}}} \tilde{\mathcal{T}}^\mu{}_\nu \dot{x}^\nu - \kappa c Z^\mu. \quad (159)$$

We substitute this P^μ into (153), (154)

$$\nabla \left[\frac{m_r}{\sqrt{-\dot{x}\mathcal{G}\dot{x}}} \tilde{\mathcal{T}}^\mu{}_\nu \dot{x}^\nu \right] = -\frac{1}{4c} \theta^\mu{}_\nu \dot{x}^\nu - \kappa \frac{\sqrt{-\dot{x}\mathcal{G}\dot{x}}}{32m_r c^2} \nabla^\mu (S\theta) + \kappa \nabla Z^\mu, \quad (160)$$

$$\nabla S^{\mu\nu} = -\frac{\kappa \sqrt{-\dot{x}\mathcal{G}\dot{x}}}{4m_r c} (\theta S)^{[\mu\nu]} - \frac{2m_r c (\kappa - 1)b}{\sqrt{-\dot{x}\mathcal{G}\dot{x}}} \dot{x}^{[\mu} (S\theta \dot{x})^{\nu]} + 2\kappa c \dot{x}^{[\mu} Z^{\nu]}. \quad (161)$$

Together with (156), this gives us the Lagrangian equations for the spinning particle with gravimagnetic moment. Comparing our equations to those of spinning particle on electromagnetic background (see Eqs. (233)-(235) below), we see that the two systems have the same structure after the identification $\kappa \sim \mu$ and $\theta_{\mu\nu} \equiv R_{\mu\nu\alpha\beta} S^{\alpha\beta} \sim F_{\mu\nu}$, where μ is the magnetic moment. That is a curvature influences trajectory of spinning particle in the same way as the electromagnetic field with the strength $\theta_{\mu\nu}$.

B. Ultra-relativistic limit requires the value of gravimagnetic moment $\kappa = 1$

In the previous subsection we have formulated Hamiltonian variational problem for spinning particle with gravimagnetic moment κ in an arbitrary gravitational background. The model is consistent for any value of κ . When $\kappa = 0$, our equations of motion (152)-(154) coincide with MPTD-equations (110). As we have shown above, they have

unsatisfactory behavior in ultra-relativistic limit. Consider now our spinning particle with gravimagnetic moment $\kappa = 1$. This implies $\tilde{T}^\mu{}_\nu = \delta^\mu{}_\nu$, $\mathcal{G}_{\mu\nu} = g_{\mu\nu}$, and crucially simplifies the equations of motion¹⁰. The Hamiltonian equations (152)-(154) read

$$\frac{m_r c}{\sqrt{-\dot{x}g\dot{x}}} \dot{x}^\mu = P^\mu + cZ^\mu, \quad (162)$$

$$\nabla P_\mu = -\frac{1}{4}\theta_{\mu\nu}\dot{x}^\nu - \frac{\sqrt{-\dot{x}g\dot{x}}}{32m_r c}\nabla_\mu(S\theta), \quad (163)$$

$$\nabla S^{\mu\nu} = -\frac{\sqrt{-\dot{x}g\dot{x}}}{4m_r c}(\theta S)^{[\mu\nu]} + 2P^{[\mu}\dot{x}^{\nu]}, \quad (164)$$

while the Lagrangian equations are composed now by the equation for trajectory

$$\nabla \left[\frac{m_r \dot{x}^\mu}{\sqrt{-\dot{x}g\dot{x}}} \right] = -\frac{1}{4c}\theta^\mu{}_\nu \dot{x}^\nu - \frac{\sqrt{-\dot{x}g\dot{x}}}{32m_r c^2}\nabla^\mu(S\theta) + \nabla Z^\mu, \quad (165)$$

and by the equation for precession of spin-tensor

$$\nabla S^{\mu\nu} = -\frac{\sqrt{-\dot{x}g\dot{x}}}{4m_r c}(\theta S)^{[\mu\nu]} + 2c\dot{x}^{[\mu}Z^{\nu]}. \quad (166)$$

These equations can be compared with (92) and (93). In the modified theory:

1. Time interval and distance should be unambiguously defined within the original space-time metric $g_{\mu\nu}$. So the critical speed is equal to the speed of light.
2. Covariant precession of spin (166) has a smooth behavior, in particular, for homogeneous field, $\nabla R = 0$, we have $\nabla S \sim \sqrt{-\dot{x}g\dot{x}}$ contrary to $\nabla S \sim \frac{1}{\sqrt{-\dot{x}g\dot{x}}}$ in the equation (93).
3. Even in homogeneous field we have modified dynamics for both x and S . The equation (165) in the space with homogeneous curvature has the structure similar to (1), hence we expect that longitudinal acceleration vanishes as $v \rightarrow c$. Let us confirm this by direct computations.

To find the acceleration, we separate derivative of the radiation mass m_r and write equation (165) in the form

$$\frac{d}{d\tau} \left[\frac{\dot{x}^\mu}{\sqrt{-\dot{x}g\dot{x}}} \right] = \frac{f^\mu}{\sqrt{-\dot{x}g\dot{x}}}, \quad (167)$$

where the force is

$$f^\mu \equiv -\Gamma_{\alpha\beta}^\mu \dot{x}^\alpha \dot{x}^\beta - \frac{\sqrt{-\dot{x}g\dot{x}}}{4m_r c}\theta^\mu{}_\nu \dot{x}^\nu + \frac{\dot{x}g\dot{x}}{32m_r^2 c^2}\nabla^\mu(S\theta) + \frac{\sqrt{-\dot{x}g\dot{x}}}{m_r}\nabla Z^\mu - \dot{x}^\mu \frac{\dot{m}_r}{m_r}. \quad (168)$$

While this expression contains derivatives of spin due to \dot{m}_r -term, the resulting expression is non singular function of velocity because ∇S is a smooth function. Hence, contrary to Eq. (125), the force now is non singular function of velocity. We take $\tau = x^0$ in the spacial part of the system (167), this gives

$$\left(\frac{dt}{dx^0} \right)^{-1} \frac{d}{dx^0} \left[\frac{v^i}{\sqrt{c^2 - (\mathbf{v}\gamma\mathbf{v})}} \right] = \frac{f^i(v)}{\sqrt{c^2 - (\mathbf{v}\gamma\mathbf{v})}}, \quad (169)$$

where $f^i(v)$ is obtained from (168) replacing \dot{x}^μ by v^μ of equation (10). This system is of the form (28), so the acceleration is given by (37) and (38)

$$a^i = \tilde{M}^i{}_j [f^j + \tilde{\Gamma}^j{}_{kl}(\gamma)v^k v^l] + \frac{1}{2} \left(\frac{dt}{dx^0} \right)^{-1} \left[(\mathbf{v}\partial_0\gamma\gamma^{-1})^i - \frac{(\mathbf{v}\partial_0\gamma\mathbf{v})}{c^2} v^i \right], \quad (170)$$

$$\mathbf{v}\gamma\mathbf{a} = \left(1 - \frac{\mathbf{v}\gamma\mathbf{v}}{c^2} \right) \left[(\mathbf{v}\gamma)_i [f^i(v) + \tilde{\Gamma}^i{}_{kl}(\gamma)v^k v^l] + \frac{1}{2} \left(\frac{dt}{dx^0} \right)^{-1} (\mathbf{v}\partial_0\gamma\mathbf{v}) \right]. \quad (171)$$

¹⁰ Besides $S^{\mu\nu}P_\nu = 0$, there are other supplementary spin conditions [3–7]. In this respect we point out that the MPTD theory implies this condition with certain P_ν written in Eq. (110). Introducing κ , we effectively changed P_ν and hence changed the supplementary spin condition. For instance, when $\kappa = 1$ and in the space with $\nabla R = 0$, we have $P^\mu = \frac{\dot{m}_c}{\sqrt{-\dot{x}g\dot{x}}} \dot{x}^\mu$ instead of (110).

With the smooth f^i given in Eq. (168), and as $v \rightarrow c$, the acceleration (170) remains finite while the longitudinal acceleration (171) vanishes. Due to the identity (35), we have $(\mathbf{v}\gamma)_i f^i \xrightarrow{v \rightarrow c} -\mathbf{v}\gamma)_i \Gamma^i_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$, that is the trajectory tends to that of spinless particle in the limit.

In resume, contrary to MPTD-equations, the modified theory is consistent with respect to the original metric $g_{\mu\nu}$. Hence the modified equations could be more promising for description of the rotating objects in astrophysics. It would be interesting to see if the non-minimal interaction allows to remove the spacelike-timelike transitions observed for spinning particle in the Schwarzschild background [22].

We should note that MPTD-equations follow from a particular form assumed for the multipole representation of a rotating body [1]. It would be interesting to find a set of multipoles which yields the modified equations (160) and (161).

C. Lagrangian of spinning particle with gravimagnetic moment

We look for the Lagrangian which in the phase space implies the variational problem (140). First, we note that the constraints $\omega\pi = P\omega = 0$ always appear from the Lagrangian which depends on $N\dot{x}$ and $N\dot{\omega}$. So we omit the corresponding terms in (140). Second, we present the remaining terms in (140) in the form

$$S_\kappa = \int d\tau \, p_\mu \dot{x}^\mu + \pi_\mu \dot{\omega}^\mu - \frac{\lambda_1}{2} (P, \pi) \begin{pmatrix} g & \lambda g \\ \lambda g & \sigma \end{pmatrix} \begin{pmatrix} P \\ \pi \end{pmatrix} - \frac{\lambda_1}{2} \left[(mc)^2 - \frac{\alpha}{\omega^2} \right], \quad (172)$$

where we have introduced the symmetric matrix

$$\sigma^{\mu\nu} = g^{\mu\nu} + \kappa R_\alpha{}^\mu{}_\beta{}^\nu \omega^\alpha \omega^\beta, \quad \text{then} \quad \sigma^{\mu\nu} \omega_\nu = \omega^\mu. \quad (173)$$

The matrix appeared in (172) is invertible, the inverse matrix is

$$\begin{pmatrix} K\sigma & -\lambda K \\ -\lambda K & K \end{pmatrix}, \quad \text{where} \quad K = (\sigma - \lambda^2 g)^{-1}. \quad (174)$$

When $\kappa = 0$ we have $K^{\mu\nu} = (1 - \lambda^2)^{-1} g^{\mu\nu}$, and (174) coincides with the matrix appeared in (52). Third, we remind that the Hamiltonian variational problem of the form $p\dot{q} - \frac{\lambda_1}{2} p A p$ follows from the reparametrization-invariant Lagrangian $\sqrt{\dot{q} A^{-1} \dot{q}}$. So, we tentatively replace the matrix appeared in the free Lagrangian (52) by (174) and switch on the minimal interaction of spin with gravity, $\dot{\omega} \rightarrow \nabla\omega$. This gives the following Lagrangian formulation of spinning particle with gravimagnetic moment:

$$L = -\sqrt{(mc)^2 - \frac{\alpha}{\omega^2}} \sqrt{-(N\dot{x}, N\nabla\omega) \begin{pmatrix} K\sigma & -\lambda K \\ -\lambda K & K \end{pmatrix} \begin{pmatrix} N\dot{x} \\ N\nabla\omega \end{pmatrix}} = \quad (175)$$

$$-\sqrt{(mc)^2 - \frac{\alpha}{\omega^2}} \sqrt{-\dot{x} N K \sigma N \dot{x} - \nabla\omega N K N \nabla\omega + 2\lambda \dot{x} N K N \nabla\omega}. \quad (176)$$

Let us show that this leads to the desired Hamiltonian formulation (140). The matrixes σ , K and N are symmetric and mutually commuting. Canonical momentum for λ vanishes and hence represents the primary constraint, $p_\lambda = 0$. In terms of the canonical momentum $P_\mu \equiv p_\mu - \Gamma_{\alpha\mu}^\beta \omega^\alpha \pi_\beta$, the expressions for conjugate momenta $p_\mu = \frac{\partial L}{\partial \dot{x}^\mu}$ and $\pi_\mu = \frac{\partial L}{\partial \dot{\omega}^\mu}$ read

$$P_\mu = \frac{1}{L_0} \left[m^2 c^2 - \frac{\alpha}{\omega^2} \right]^{\frac{1}{2}} [(\dot{x} N K \sigma N)_\mu - \lambda (\nabla\omega N K N)_\mu], \quad (177)$$

$$\pi_\mu = \frac{1}{\sqrt{2} L_0} \left[m^2 c^2 - \frac{\alpha}{\omega^2} \right]^{\frac{1}{2}} [(\nabla\omega N K N)_\mu - \lambda (\dot{x} N K N)_\mu], \quad (178)$$

where L_0 is the second square root in (176). They immediately imply the primary constraints $\omega\pi = 0$ and $P\omega = 0$. From the expressions

$$P^2 = \frac{1}{L_0^2} \left[(mc)^2 - \frac{\alpha}{\omega^2} \right] [(\dot{x} N K \sigma K \sigma N \dot{x}) + \lambda^2 (\nabla\omega N K K N \nabla\omega) - 2\lambda (\dot{x} N K \sigma K N \nabla\omega)],$$

$$\begin{aligned}
\pi\sigma\pi &= \frac{1}{L_0^2} \left[(mc)^2 - \frac{\alpha}{\omega^2} \right] \left[\lambda^2 (\dot{x}NK\sigma KN\dot{x}) + (\nabla\omega NK\sigma KN\nabla\omega) - 2\lambda(\dot{x}NK\sigma KN\nabla\omega) \right], \\
2\lambda P\pi &= \frac{1}{L_0^2} \left[(mc)^2 - \frac{\alpha}{\omega^2} \right] \left[-2\lambda^2 (\dot{x}NK\sigma KN\dot{x}) - 2\lambda^2 (\nabla\omega NKKN\nabla\omega) + 2\lambda(\dot{x}NK\sigma KN\nabla\omega) + \right. \\
&\quad \left. 2\lambda^3 (\dot{x}NKKN\nabla\omega) \right], \tag{179}
\end{aligned}$$

we conclude that their sum does not depend on velocities and hence gives one more constraint

$$P^2 + \pi\sigma\pi + 2\lambda P\pi = - \left[(mc)^2 - \frac{\alpha}{\omega^2} \right]. \tag{180}$$

Then the Hamiltonian is $H = P\dot{x} + \pi\nabla\omega - L + \lambda_i T_i$. From (177) and (178) we obtain $P\dot{x} + \pi\nabla\omega = L$, so the Hamiltonian is composed from primary constraints

$$H = \frac{\lambda_1}{2} \left[P^2 + \kappa R_{\alpha\mu\beta\nu} \omega^\alpha \pi^\mu \omega^\beta \pi^\nu + (mc)^2 + \pi^2 - \frac{\alpha}{\omega^2} + 2\lambda(P\pi) \right] + \lambda_2(\omega\pi) + \lambda_3(P\omega) + v p_\lambda. \tag{181}$$

This expression is equivalent to the Hamiltonian written in the variational problem (140). The problems (140) and (181) yield the same equations for the set x^μ , P^μ and $S^{\mu\nu}$.

VI. SPINLESS PARTICLE NON-MINIMALLY INTERACTING WITH ELECTROMAGNETIC FIELD, SPEED OF LIGHT AND CRITICAL SPEED

In this section we consider examples of manifestly Poincare-covariant and reparametrization-invariant equations of spinless particle in Minkowski space which lead to critical speed different from the speed of light. We achieve this assuming a non-minimal interaction with electromagnetic field. These toy models confirm that the critical speed different from the speed of light does not contradict relativistic invariance (existence of the observer-independent scale c).

Let us denote $x^i(t)$, $i = 1, 2, 3$ physical dynamical variables describing trajectory of relativistic particle subject to an external force. In order to work with manifestly covariant quantities, we use parametric equations of the trajectory $x^\mu(\tau) = (x^0 \equiv ct(\tau), x^i(\tau))$, where τ is an arbitrary parameter along the world line. In this section we use the usual special-relativity notions for time, three-dimensional distance, velocity and acceleration as well as for the scalar product

$$v^i = \frac{dx^i}{dt}, \quad a^i = \frac{dv^i}{dt}, \quad \mathbf{va} = v^i a^i. \tag{182}$$

Let us start from the standard Lagrangian of spinless particle in electromagnetic field. Using the auxiliary variable λ , the Lagrangian reads

$$L = \frac{1}{2\lambda} \dot{x}^2 - \frac{\lambda}{2} m^2 c^2 + \frac{e}{c} A \dot{x}. \tag{183}$$

This implies the manifestly relativistic equations

$$\left(\frac{\dot{x}^i}{\sqrt{-\dot{x}^2}} \right)' = \frac{e}{mc^2} F^\mu{}_\nu \dot{x}^\nu. \tag{184}$$

They became singular as $\dot{x}^2 \rightarrow 0$. Using the reparametrization-invariant derivative $D = \frac{1}{\sqrt{-\dot{x}^2}} \frac{d}{d\tau}$, they read in manifestly reparametrization-invariant form

$$DDx^\mu = f^\mu \equiv \frac{e}{mc^2} F^\mu{}_\nu Dx^\nu. \tag{185}$$

Due to the identities

$$\dot{x}_\mu DDx^\mu = 0, \quad \dot{x}_\mu f^\mu = 0, \tag{186}$$

the system contains only three independent equations. The first identity became more transparent if we compute derivative on the left hand side of (184), then the system reads

$$M^\mu{}_\nu \ddot{x}^\nu = \frac{e\sqrt{-\dot{x}^2}}{mc^2} F^\mu{}_\nu \dot{x}^\nu, \tag{187}$$

where $M^\mu{}_\nu$ turns out to be projector on the plane orthogonal to \dot{x}^μ

$$M^\mu{}_\nu = \delta^\mu{}_\nu - \frac{\dot{x}^\mu \dot{x}_\nu}{\dot{x}^2}, \quad \dot{x}_\mu M^\mu{}_\nu = 0. \quad (188)$$

Using reparametrization invariance, we can take physical time as the parameter, $\tau = t$, this directly yields equations for observable dynamical variables $x^i(t)$. In the physical-time parametrization we have $x^\mu = (ct, \mathbf{x}(t))$, $\dot{x}^\mu = (c, \mathbf{v}(t))$ and $\frac{1}{\sqrt{-\dot{x}^2}} = \frac{1}{\sqrt{c^2 - \mathbf{v}^2}}$. Time-like component of the system (184) reads

$$\frac{d}{dt} \frac{c}{\sqrt{c^2 - \mathbf{v}^2}} = \frac{e}{mc^2} F^0{}_i \dot{x}^i. \quad (189)$$

and gives the value of acceleration along the direction of velocity

$$\mathbf{v}\mathbf{a} = \frac{e(c^2 - \mathbf{v}^2)^{\frac{3}{2}}}{mc^3} (\mathbf{E}\mathbf{v}). \quad (190)$$

The longitudinal acceleration vanishes as $|\mathbf{v}| \rightarrow c$. Hence the singularity in Eq. (184) implies that the particles speed can not exceed the value c .

Components of three-acceleration vector can be obtained from the spatial part of the system (184)

$$\frac{\ddot{x}^i}{\sqrt{c^2 - \mathbf{v}^2}} + \dot{x}^i \frac{d}{dt} \frac{1}{\sqrt{c^2 - \mathbf{v}^2}} = \frac{e}{mc^2} F^i{}_\nu \dot{x}^\nu. \quad (191)$$

Using Eq. (189) we obtain

$$\mathbf{a} = \frac{\sqrt{c^2 - \mathbf{v}^2}}{mc} \left[e\mathbf{E} - \frac{e(\mathbf{E}\mathbf{v})}{c^2} \mathbf{v} + \frac{e}{c} \mathbf{v} \times \mathbf{B} \right]. \quad (192)$$

In accordance with degeneracy (186), scalar product of the spacial part with \mathbf{v} gives the time-like component, Eq. (190).

Let us discuss two modifications which preserve both relativistic covariance and reparametrization invariance of equations of motion, but could yield non vanishing longitudinal acceleration as $|\mathbf{v}| \rightarrow c$.

First, in the presence of external fields we can construct an additional reparametrization invariants. For instance, we can use the derivative

$$D' \equiv \frac{1}{\sqrt{-\dot{x}g\dot{x}}} \frac{d}{d\tau}, \quad (193)$$

where the usual relativistic factor $\dot{x}\eta\dot{x}$ is replaced by

$$-\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu = -\dot{x}^2 - \epsilon k (\dot{x} F F \dot{x}) = c^2 - \mathbf{v}^2 - \epsilon k (\dot{x} F F \dot{x}), \quad \epsilon = \pm 1. \quad (194)$$

The right dimension can be supplied by the constant k equal to $\frac{e^6}{m^4 c^8}$ or $\frac{\hbar^3}{m^4 c^5}$. Second, we consider non minimal interaction with the force constructed from reparametrization-invariant quantities, $f^\mu(D'x, F, \partial F, \dots) = f^\mu{}_0 + f^\mu{}_\nu D'x^\nu + \dots$. Hence let us consider the manifestly covariant and reparametrization-invariant equations

$$D'D'x^\mu = f^\mu, \quad \text{or} \quad M^\mu{}_\nu \ddot{x}^\nu = -(\dot{x}g\dot{x})f^\mu + \frac{(\dot{x}\dot{g}\dot{x})}{2(\dot{x}g\dot{x})} \dot{x}^\mu, \quad (195)$$

where

$$M^\mu{}_\nu = \delta^\mu{}_\nu - \frac{\dot{x}^\mu (\dot{x}g)_\nu}{(\dot{x}g\dot{x})}, \quad (\dot{x}g)_\mu M^\mu{}_\nu = 0, \quad M^\mu{}_\nu \dot{x}^\nu = 0. \quad (196)$$

Due to non invertibility of $M^\mu{}_\nu$, the system (195) consist of three second-order equations and one equation of first order. Contracting (195) with $(\dot{x}g)_\mu$, we obtain the first-order equation

$$\dot{x}\dot{g}\dot{x} = 2(\dot{x}g\dot{x})(\dot{x}gf). \quad (197)$$

This is analog of $\dot{x}_\mu f^\mu = 0$ of the previous case (186), and can be used to present f^0 through three-dimensional force. Using reparametrization invariance, we take $\tau = x^0$ and write (195) in the form

$$\frac{d}{dx^0} \frac{1}{\sqrt{-\dot{x}g\dot{x}}} = \sqrt{-\dot{x}g\dot{x}} f^0, \quad (198)$$

$$\frac{\ddot{x}^i}{\sqrt{-\dot{x}g\dot{x}}} + \dot{x}^i \frac{d}{dx^0} \frac{1}{\sqrt{-\dot{x}g\dot{x}}} = \sqrt{-\dot{x}g\dot{x}} f^i. \quad (199)$$

Using (198) in (199) we obtain

$$\frac{d^2 x^i}{dx^{02}} = -\dot{x}g\dot{x} \left[f^i - f^0 \frac{dx^i}{dx^0} \right], \quad \dot{x}^\mu = \frac{dx^\mu}{dx^0}. \quad (200)$$

We take $x^0 = ct$, then three-acceleration

$$\mathbf{a} = -\dot{x}g\dot{x} \left[\mathbf{f} - \frac{f^0}{c} \mathbf{v} \right], \quad \dot{x}^\mu = (c, \mathbf{v} = \frac{d\mathbf{x}}{dt}). \quad (201)$$

Equations (197), (201) are equivalent to the initial system (195). Eq. (201) implies the longitudinal acceleration

$$\mathbf{v}\mathbf{a} = -\dot{x}g\dot{x} \left[\mathbf{v}\mathbf{f} - \frac{\mathbf{v}^2}{c} f^0 \right] \equiv -\dot{x}g\dot{x} \left[\frac{c^2 - \mathbf{v}^2}{c} f^0 + \dot{x}_\mu f^\mu \right], \quad f^\mu = f^\mu \left(\frac{\dot{x}}{\sqrt{-\dot{x}g\dot{x}}}, F, \partial F, \dots \right). \quad (202)$$

The acceleration vanishes at the values of speed which zero out r.h.s. of this equation. If in physical-time parametrization the four-force obeys the identity $\dot{x}_\mu f^\mu = 0$, we have two special points, $|\mathbf{v}| = c$, and v' determined from $\dot{x}g\dot{x} = 0$. In absence of the identity, and if $\dot{x}_\mu f^\mu \neq 0$ as $|\mathbf{v}| \rightarrow c$, the speed of light does not represent a special point of (202). Let us illustrate this equation with two examples.

Example 1. Consider the minimal interaction $f^\mu = \frac{e}{mc^2} F^\mu{}_\nu D'x^\nu$ and the relativistic-contraction factor (194), then (202) reads

$$\mathbf{v}\mathbf{a} = \frac{e(\mathbf{v}\mathbf{E})}{mc^3} \sqrt{-\dot{x}g\dot{x}} (c^2 - \mathbf{v}^2). \quad (203)$$

Besides the usual special point, $\mathbf{v}^2 = c^2$, there is one more, say $v' = |\mathbf{v}'|$, determined by $\dot{x}g\dot{x} = 0$. This surface is slightly different from the sphere $c^2 - \mathbf{v}^2 = 0$. So the second special point generally differs from the speed of light. To see this, we compute the last term in (194)

$$-\dot{x}FF\dot{x} = c^2 E_i \left(\delta_{ij} - \frac{v_i v_j}{c^2} \right) E_j + \mathbf{v}^2 B_i N_{ij} B_j. \quad (204)$$

Here $N_{ij} \equiv \delta_{ij} - \frac{v_i v_j}{\mathbf{v}^2}$ is projection operator on the plane orthogonal to the vector \mathbf{v} , so we can write $\mathbf{B}N\mathbf{B} = (N\mathbf{B})^2 = \mathbf{B}_\perp^2$. Then the factor (194) reads

$$-\dot{x}g\dot{x} = c^2 - \mathbf{v}^2 + \epsilon k \left[c^2 \mathbf{E} \left(1 - \frac{\mathbf{v}\mathbf{v}}{c^2} \right) \mathbf{E} + \mathbf{v}^2 \mathbf{B}_\perp^2 \right]. \quad (205)$$

The quantity $\delta_{ij} - \frac{v_i v_j}{c^2}$ turns into the projection operator N when $|\mathbf{v}| = c$. Hence

$$-\dot{x}g\dot{x} \xrightarrow{|\mathbf{v}| \rightarrow c} \epsilon k c^2 [\mathbf{E}_\perp^2 + \mathbf{B}_\perp^2]. \quad (206)$$

If \mathbf{E} and \mathbf{B} are not mutually parallel in the laboratory system, this expression does not vanish for any orientation of \mathbf{v} . This implies that the factor (194) does not vanish at $|\mathbf{v}| = c$.

We confirmed that longitudinal acceleration generally vanishes at two different values of speed, c and v' . Then Eq. (203) implies the following possibilities.

A) Let $\epsilon = +1$, then from (205) we conclude $c < v'$, and speed of the particle approximates to c . The second special point v' turns out to be irrelevant. So $v_{cr} = c$.

B) Let $\epsilon = -1$, then $v' < c$, and the particle with small initial velocity will approximate to the critical velocity $v_{cr} = v' < c$. So it never approximates to the speed of light.

Example 2. Possibility of superluminal motion? Take the relativistic-contraction factor (194) with $\epsilon = +1$ and non-parallel electric and magnetic fields. As we have seen above, this implies $c < v'$, where v' is a solution of $\dot{x}g\dot{x} = 0$. Consider the non-minimal interaction

$$f^\mu = \frac{e}{mc^2} F^\mu{}_\nu D'x^\nu + \tilde{f}^\mu, \quad \text{where} \quad \tilde{f}^\mu = -\tilde{k}^2 D'x^\alpha \partial^\mu (FF)_{\alpha\beta} D'x^\beta. \quad (207)$$

We assume homogeneous and non-stationary fields with growing tension

$$\partial^i \mathbf{E} = \partial^i \mathbf{B} = 0, \quad \frac{d}{dt} |\mathbf{E}| > 0, \quad \frac{d}{dt} |\mathbf{B}| > 0, \quad (208)$$

then $\tilde{f}^i = 0$, $\tilde{f}^0 = -\frac{\tilde{k}^2}{c\dot{x}g\dot{x}}\dot{x}\frac{\partial}{\partial t}(FF)\dot{x}$. The longitudinal acceleration reads

$$\mathbf{va} = a_1(v) + a_2(v) \equiv \frac{e(\mathbf{vE})}{mc^3}\sqrt{-\dot{x}g\dot{x}}(c^2 - \mathbf{v}^2) - \frac{\tilde{k}^2\mathbf{v}^2}{c^2}\dot{x}\frac{\partial}{\partial t}(FF)\dot{x}. \quad (209)$$

We have $a_1(c) = 0$, while $a_2(c)$ is positive according to Eqs. (204) and (208). So the particle overcomes the light barrier. In the region $c < v < v'$ we have $a_1(v) < 0$ and $a_2(v) > 0$, so the particle will continue to accelerate up to critical velocity v_{cr} determined by the equation $a_1 + a_2 = 0$. If $a_2 > |a_1|$ in the region, the particle will accelerate up to the value $v_{cr} = v'$. Above this velocity, the equation (209) becomes meaningless.

The toy examples show that critical speed in a manifestly relativistic and reparametrization-invariant theory does not always coincide with the speed of light, if we assume the usual special-relativity definitions of time and distance. In general case, we expect that \mathbf{v}_{cr} is both field and spin-dependent quantity. In the next section we repeat this analysis for more realistic case of a particle with spin.

VII. SPINNING PARTICLE IN AN ARBITRARY ELECTROMAGNETIC FIELD

A. Lagrangian and Hamiltonian formulations

In the formulation (50), the vector model of spin admits interaction with an arbitrary electromagnetic field. To introduce coupling of the position variable x with electromagnetic field we add the minimal interaction $\frac{e}{c}A_\mu\dot{x}^\mu$. As for spin, it couples with A^μ through the term

$$D\omega^\mu \equiv \dot{\omega}^\mu - \lambda \frac{e\mu}{c}F^{\mu\nu}\omega_\nu, \quad (210)$$

where the coupling constant μ is the magnetic moment. They are the only terms we have found compatible with symmetries and constraints which should be presented in the theory. Adding these terms to the free theory (50) we obtain the action

$$S = \int d\tau \frac{1}{4\lambda} \left[\dot{x}N\dot{x} + D\omega ND\omega - \sqrt{[\dot{x}N\dot{x} + D\omega ND\omega]^2 - 4(\dot{x}ND\omega)^2} \right] - \frac{\lambda}{2} \left(m^2 c^2 - \frac{\alpha}{\omega^2} \right) + \frac{e}{c} A\dot{x}. \quad (211)$$

In their work [20], Hanson and Regge analyzed whether the spin-tensor (49) interacts directly with an electromagnetic field, and concluded on impossibility to construct the interaction in closed form. In our model an electromagnetic field interacts with the part ω^μ of the spin-tensor.

Let us construct Hamiltonian formulation of the model. The procedure which leads to the Hamiltonian turns out to be very similar to that described in subsection IV A, so we present only the final expression (for the details, see [29]). Conjugate momenta for x^μ , ω^μ and λ are denoted as p^μ , π^μ and p_λ . We use also the canonical momentum $\mathcal{P}^\mu \equiv p^\mu - \frac{e}{c}A^\mu$. Contrary to p^μ , the canonical momentum \mathcal{P}^μ is $U(1)$ gauge-invariant quantity. With these notation, we obtain the Hamiltonian variational problem which is equivalent to (211)

$$S_H = \int d\tau \left[p_\mu \dot{x}^\mu + \pi_\mu \dot{\omega}^\mu + p_\lambda \dot{\lambda} - \left[\frac{\lambda}{2} \left(P^2 - \frac{e\mu}{2c}(FS) + (mc)^2 + \pi^2 - \frac{\alpha}{\omega^2} \right) + \lambda_2(\omega\pi) + \lambda_3(P\omega) + \lambda_4(P\pi) + \lambda_0 p_\lambda \right] \right] \quad (212)$$

The expression in square brackets represents the Hamiltonian. By λ_2 , λ_3 , λ_4 and λ_0 we denoted the Lagrangian multipliers, they are written in front of the corresponding primary constraints. The fundamental Poisson brackets $\{x^\mu, p^\nu\} = \eta^{\mu\nu}$ and $\{\omega^\mu, \pi^\nu\} = \eta^{\mu\nu}$ imply

$$\{x^\mu, \mathcal{P}^\nu\} = \eta^{\mu\nu}, \quad \{\mathcal{P}^\mu, \mathcal{P}^\nu\} = \frac{e}{c}F^{\mu\nu}, \quad (213)$$

$$\{S^{\mu\nu}, S^{\alpha\beta}\} = 2(\eta^{\mu\alpha}S^{\nu\beta} - \eta^{\mu\beta}S^{\nu\alpha} - \eta^{\nu\alpha}S^{\mu\beta} + \eta^{\nu\beta}S^{\mu\alpha}). \quad (214)$$

According to Eq. (214) the spin-tensor is generator of Lorentz algebra $SO(1,3)$. As $\omega\pi$, ω^2 and π^2 are Lorentz-invariants, they have vanishing Poisson brackets with $S^{\mu\nu}$. To reveal the higher-stage constraints we write the

equations $\dot{T}_i = \{T_i, H\} = 0$. The Dirac procedure stops on third stage with the following equations

$$p_\lambda = 0 \quad \Rightarrow \quad T_1 \equiv \mathcal{P}^2 - \frac{e\mu}{2c}(FS) + m^2c^2 + \pi^2 - \frac{\alpha}{\omega^2} = 0 \quad \Rightarrow \quad \lambda_3 C + \lambda_4 D = 0, \quad (215)$$

$$T_2 \equiv (\omega\pi) = 0 \quad \Rightarrow \quad T_5 \equiv \pi^2 - \frac{\alpha}{\omega^2} = 0, \quad (216)$$

$$T_3 \equiv (\mathcal{P}\omega) = 0 \quad \Rightarrow \quad \lambda_4 = -\frac{2\lambda c}{e}aC, \quad (217)$$

$$T_4 \equiv (\mathcal{P}\pi) = 0 \quad \Rightarrow \quad \lambda_3 = \frac{2\lambda c}{e}aD. \quad (218)$$

We have denoted

$$C = -\frac{e(\mu-1)}{c}(\omega F\mathcal{P}) + \frac{e\mu}{4c}(\omega\partial)(FS), \quad D = -\frac{e(\mu-1)}{c}(\pi F\mathcal{P}) + \frac{e\mu}{4c}(\pi\partial)(FS). \quad (219)$$

Besides, here and below we will use the following notation. In the equation which relates velocity and canonical momentum will appear the matrix $T^{\mu\nu}$

$$T^{\mu\nu} = \eta^{\mu\nu} - (\mu-1)a(SF)^{\mu\nu}, \quad a = \frac{-2e}{4m^2c^3 - e(2\mu+1)(SF)}. \quad (220)$$

Using the identity $S^{\mu\alpha}F_{\alpha\beta}S^{\beta\nu} = -\frac{1}{2}(S^{\alpha\beta}F_{\alpha\beta})S^{\mu\nu}$ we find the inverse matrix

$$\tilde{T}^{\mu\nu} = \eta^{\mu\nu} + (\mu-1)b(SF)^{\mu\nu}, \quad b = \frac{2a}{2 + (\mu-1)a(SF)} \equiv \frac{-2e}{4m^2c^3 - 3e\mu(SF)}, \quad (221)$$

The vector Z^μ is defined by

$$Z^\mu = \frac{b}{4c}S^{\mu\sigma}(\partial_\sigma F_{\alpha\beta})S^{\alpha\beta} \equiv \frac{b}{4c}S^{\mu\sigma}\partial_\sigma(FS). \quad (222)$$

This vanishes for homogeneous field, $\partial F = 0$. The last equation from (215) turns out to be a consequence of (217) and (218) and can be omitted. Due to the secondary constraint T_5 appeared in (216) we can replace the constraint T_1 on the equivalent one

$$T_1 \equiv \mathcal{P}^2 - \frac{e\mu}{2c}(FS) + m^2c^2 = 0. \quad (223)$$

The Dirac procedure revealed two secondary constraints written in Eqs. (223) and (216), and fixed the Lagrangian multipliers λ_3 and λ_4 , the latter can be substituted into the Hamiltonian. The multipliers λ_0 , λ_2 and the auxiliary variable λ have not been determined. H vanishes on the complete constraint surface, as it should be in a reparametrization-invariant theory.

We summarized the algebra of Poisson brackets between constraints in the Table II. The constraints p_λ , T_1 , T_2 and T_5 form the first-class subset, while T_3 and T_4 represent a pair of second class. The presence of two primary first-class constraints p_λ and T_2 is in correspondence with the fact that two lagrangian multipliers remain undetermined within the Dirac procedure.

The evolution of the basic variables obtained according the standard rule $\dot{z} = \{z, H\}$ (equivalently, we can look for the extremum of the variational problem (212)). The equations read

$$\dot{x}^\mu = \lambda(T^\mu{}_\nu \mathcal{P}^\nu + \frac{\mu c a}{b}Z^\mu), \quad \dot{\mathcal{P}}^\mu = \frac{e}{c}(F\dot{x})^\mu + \lambda \frac{\mu e}{4c}\partial^\mu(FS), \quad (224)$$

$$\begin{aligned} \dot{\omega}^\mu &= \lambda \frac{e\mu}{c}(F\omega)^\mu - \lambda \frac{2caC}{e}\mathcal{P}^\mu + \pi^\mu + \lambda_2\omega^\mu, \\ \dot{\pi}^\mu &= \lambda \frac{e\mu}{c}(F\pi)^\mu - \lambda \frac{2caD}{e}\mathcal{P}^\mu - \frac{\alpha}{(\omega^2)^2}\omega^\mu - \lambda_5 2\pi^\mu, \end{aligned} \quad (225)$$

The ambiguity due to the variables λ and λ_2 means that the interacting theory preserves both reparametrization and spin-plane symmetries of the free theory. As a consequence, all the basic variables have ambiguous evolution. x^μ and \mathcal{P}^μ have one-parametric ambiguity due to λ while ω and π have two-parametric ambiguity due to λ and λ_2 . The

	T_1	T_5	T_2	T_3	T_4
$T_1 = \mathcal{P}^2 - \frac{\mu e}{2c}(FS) + m^2 c^2$	0	0	0	-2C	-2D
$T_5 = \pi^2 - \frac{\alpha}{\omega^2}$	0	0	$-2T_5 \approx 0$	$-2T_4 \approx 0$	$\frac{2\alpha}{(\omega^2)^2} T_3 \approx 0$
$T_2 = \omega\pi$	0	$2T_5 \approx 0$	0	$-T_3 \approx 0$	$T_4 \approx 0$
$T_3 = \mathcal{P}\omega$	2C	$2T_4 \approx 0$	$T_3 \approx 0$	0	$T_1 + \frac{e}{2ca}$ $\approx \frac{e}{2ca}$
$T_4 = \mathcal{P}\pi$	2D	$-\frac{2\alpha}{(\omega^2)^2} T_3 \approx 0$	$-T_4 \approx 0$	$-T_1 - \frac{e}{2ca}$ $\approx -\frac{e}{2ca}$	0

TABLE II: Algebra of constraints.

variables with ambiguous dynamics do not represent observable quantities, so we look for the variables that can be candidates for observable quantities. We note that (225) imply an equation for $S^{\mu\nu}$ which does not contain λ_2

$$\dot{S}^{\mu\nu} = \lambda \frac{e\mu}{c} (FS)^{[\mu\nu]} + 2\mathcal{P}^{[\mu} \dot{x}^{\nu]}. \quad (226)$$

This proves that the spin-tensor is invariant under local spin-plane symmetry. The remaining ambiguity due to λ contained in Eqs. (224) and (226) is related with reparametrization invariance and disappears when we work with physical dynamical variables $x^i(t)$. Thus we will work with x^μ , \mathcal{P}^μ and $S^{\mu\nu}$.

The term $\frac{\alpha}{2\omega^2}$ in the Lagrangian (211) provides the constraint T_5 which can be written as follows: $\omega^2 \pi^2 = \alpha$. Together with $\omega\pi = 0$, this implies fixed value of spin

$$S^{\mu\nu} S_{\mu\nu} = 8(\omega^2 \pi^2 - (\omega\pi)^2) = 8\alpha, \quad (227)$$

for any solution to the equations of motion. The constraints $\omega\mathcal{P} = \pi\mathcal{P} = 0$ imply the Pirani condition for the spin-tensor (49)

$$S^{\mu\nu} \mathcal{P}_\nu = 0. \quad (228)$$

The equations (227) and (228) imply that only two components of spin-tensor are independent, as it should be for spin one-half particle.

Equations (224) and (226), together with (227) and (228), form a closed system which determines evolution of a spinning particle.

The quantities x^μ , \mathcal{P}^μ and $S^{\mu\nu}$, being invariant under spin-plane symmetry, have vanishing brackets with the corresponding first-class constraints T_2 and T_5 . So, obtaining equations for these quantities, we can omit the corresponding terms in the Hamiltonian (212). Further, we can construct the Dirac bracket for the second-class pair T_3 and T_4 . Since the Dirac bracket of a second-class constraint with any quantity vanishes, we can now omit T_3 and T_4 from (212). Then the relativistic Hamiltonian acquires an expected form

$$H = \frac{\lambda}{2} \left(\mathcal{P}^2 - \frac{e\mu}{2c} (FS) + m^2 c^2 \right). \quad (229)$$

The equations (224) and (226) follow from this H with use of Dirac bracket, $\dot{z} = \{z, H\}_{DB}$.

We can exclude the momenta \mathcal{P} and the auxiliary variable λ from the equations of motion. This yields second-order equation for the particle's position. To achieve this, we solve the first equation from (224) with respect to \mathcal{P} and use the identities $(SFZ)^\mu = -\frac{1}{2}(SF)Z^\mu$, $\tilde{T}^\mu{}_\nu Z^\nu = \frac{b}{a}Z^\mu$, this gives $\mathcal{P}^\mu = \frac{1}{\lambda} \tilde{T}^\mu{}_\nu \dot{x}^\nu - \mu c Z^\mu$. Then the Pirani condition reads $\frac{1}{\lambda} (S\tilde{T}\dot{x})^\mu = \mu c (SZ)^\mu$. Using this equality, \mathcal{P}^2 can be presented as $\mathcal{P}^2 = \frac{1}{\lambda^2} (\dot{x}G\dot{x}) + \mu^2 c^2 Z^2$, where appeared the symmetric matrix

$$G_{\mu\nu} = (\tilde{T}^T \tilde{T})_{\mu\nu} = [\eta + b(\mu - 1)(SF + FS) + b^2(\mu - 1)^2 FSSF]_{\mu\nu}. \quad (230)$$

The matrix G is composed from the Minkowsky metric $\eta_{\mu\nu}$ plus (spin and field-dependent) contribution, $G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(S)$. So we call G the effective metric produced along the world-line by interaction of spin with electromagnetic

field. We substitute \mathcal{P}^2 into the constraint (223), this gives expression for λ

$$\lambda = \frac{\sqrt{-\dot{x}G\dot{x}}}{m_r c}, \quad m_r^2 = m^2 - \frac{\mu e}{2c^3}(FS) - \mu^2 Z^2. \quad (231)$$

This shows that the presence of λ in Eq. (210) implies highly non-linear interaction of spinning particle with electromagnetic field. The final expression of canonical momentum through velocity is

$$\mathcal{P}^\mu = \frac{m_r c}{\sqrt{-\dot{x}G\dot{x}}} \tilde{T}^\mu{}_\nu \dot{x}^\nu - \mu c Z^\mu. \quad (232)$$

Using (231) and (232), we exclude \mathcal{P}^μ and λ from the Hamiltonian equations (224), (226) and (228). This gives closed system of Lagrangian equations for the set x, S . We have the dynamical equations

$$D \left[m_r (\tilde{T} D x)^\mu \right] = \frac{e}{c^2} (F D x)^\mu + \frac{\mu e}{4 m_r c^3} \partial^\mu (S F) + \mu D Z^\mu, \quad (233)$$

$$D S^{\mu\nu} = \frac{e\mu}{m_r c^2} (F S)^{[\mu\nu]} - 2 b m_r c (\mu - 1) D x^{[\mu} (S F D x)^{\nu]} + 2 \mu c D x^{[\mu} Z^{\nu]}, \quad (234)$$

the Lagrangian counterpart of Pirani condition

$$S^{\mu\nu} [(\tilde{T} \dot{x})_\nu - \frac{\mu \sqrt{-\dot{x}G\dot{x}}}{m_r} Z_\nu] = 0, \quad (235)$$

as well as to the value-of-spin condition, $S^{\mu\nu} S_{\mu\nu} = 8\alpha$. In the approximation $O^3(S, F, \partial F)$ and when $\mu = 1$ they coincide with Frenkel equations, see [28].

Eq. (233) shows how spin modifies the Lorentz-force equation (1). In general case, the Lorentz force is modified due to the presence of (time-dependent) radiation mass m_r (231), the tetrad field \tilde{T} , the effective metric G and due to two extra-terms on right hand side of (233). Contribution of anomalous magnetic moment $\mu \neq 1$ to the difference between \dot{x}^μ and \mathcal{P}^μ in (232) is proportional to $\frac{J}{c^3} \sim \frac{\hbar}{c^3}$, while the term with a gradient of field is proportional to $\frac{J^2}{c^3} \sim \frac{\hbar^2}{c^3}$.

Consider the homogeneous field,

$$\partial_\alpha F^{\mu\nu} = 0, \quad Z^\mu = 0. \quad (236)$$

Then contraction of (235) with $F_{\mu\nu}$ yields $(SF) = 0$, that is $S^{\mu\nu} F_{\mu\nu}$ turns out to be the conserved quantity. This implies $\dot{m}_r = \dot{a} = \dot{b} = 0$. Hence the Lorentz force is modified due to the presence of time-independent radiation mass m_r , the tetrad field \tilde{T} and the effective metric G .

Consider the “classical” value of magnetic moment $\mu = 1$. Then $\tilde{T}^{\mu\nu} = \eta^{\mu\nu}$ and $G_{\mu\nu} = \eta_{\mu\nu}$. The Lorentz force is modified due to the presence of time-dependent radiation mass m_r , and two extra-terms on right hand side of (233).

Let us specify the equation for spin precession to the case of uniform and stationary field, supposing also $\mu = 1$ and taking physical time as the parameter, $\tau = t$. Then (235) reduces to the Frenkel condition, $S^{\mu\nu} \dot{x}_\nu = 0$, while (234) reads $\dot{S}^{\mu\nu} = \frac{e\sqrt{-\dot{x}^2}}{m_r c^2} (F S)^{[\mu\nu]}$. We decompose spin-tensor on electric dipole moment \vec{D} and Frenkel spin-vector \vec{S} according to (49), then $\vec{D} = -\frac{2}{c} \vec{S} \times \vec{v}$, while precession of \vec{S} is given by

$$\frac{d\vec{S}}{dt} = \frac{e\sqrt{c^2 - \vec{v}^2}}{m_r c^3} \left[-\vec{E} \times (\vec{v} \times \vec{S}) + c \vec{S} \times \vec{B} \right]. \quad (237)$$

B. Ultra-relativistic limit within the usual special-relativity notions

After identification $\theta_{\mu\nu} \equiv R_{\mu\nu\alpha\beta} S^{\alpha\beta} \sim F_{\mu\nu}$ and $\kappa \sim \mu$, equations of motion in electromagnetic and in gravitational fields acquire the similar structure. Equations (224) and (226) can be compared with (152)-(154) and (233)-(234) with (160), (161). In particular, in the Lagrangian equations with anomalous magnetic moment ($\mu \neq 1$) in Minkowski space also appeared the effective metric (230). So we need to examine the ultra-relativistic limit. In this section we do this under the usual special-relativity notions, that is we suppose that the particle probes the three-dimensional geometry (182). We show that the critical speed turns out to be different from the speed of light while an acceleration, contrary to subsection IV D, vanishes in ultra-relativistic limit. It will be sufficient to estimate the acceleration in the

uniform and stationary field (236). We take $\tau = t$ in equations (233)-(235) and compute the time derivative on l. h. s. of equations (233) with $\mu = 1, 2, 3$. Then the equations read

$$a^i - \frac{v^i}{2(-vGv)} \frac{d}{dt}(-vGv) = T^i{}_\nu \left[\frac{e\sqrt{-vGv}}{m_r c^2} (Fv)^i - \frac{d}{dt} \tilde{T}^\nu{}_\alpha v^\alpha \right], \quad (238)$$

$$\frac{d}{dt} S^{\mu\nu} = \frac{e\mu\sqrt{-vGv}}{m_r c^2} (FS)^{[\mu\nu]} - \frac{2bm_r c(\mu-1)}{\sqrt{-vGv}} v^{[\mu} (SFv)^{\nu]}, \quad (239)$$

$$(Sv)^\mu + b(\mu-1)(SSFv)^\mu = 0, \quad (240)$$

where $v^\mu = (c, \mathbf{v})$. Eqs. (240) and (230) imply

$$-vGv = -v\tilde{T}v = c^2 - \mathbf{v}^2 - (\mu-1)b(vSFv). \quad (241)$$

We compute the time-derivatives in Eq. (238)

$$\begin{aligned} \frac{d}{dt}(-vGv) = -2(\mathbf{v}\mathbf{a}) - (\mu-1)b \left\{ [v(FS + SF)]_i a^i + \frac{e\mu\sqrt{-vGv}}{m_r c^2} [(vFFSv) + (vFSFv)] - \right. \\ \left. \frac{2bm_r c(\mu-1)}{\sqrt{-vGv}} [v^2(vFSFv) - (vSFv)(vFv)] \right\}, \end{aligned} \quad (242)$$

$$\begin{aligned} -T^i{}_\nu \frac{d}{dt} \tilde{T}^\nu{}_\alpha v^\alpha = -\frac{e\sqrt{-vGv}}{m_r c^2} \left\{ \mu(\mu-1)b(FSFv)^i - \mu(\mu-1)a(SFFv)^i - \mu(\mu-1)^2 ab(SFFSFv)^i \right\} + \\ \frac{2bm_r c(\mu-1)}{\sqrt{-vGv}} T^i{}_\nu [v^\nu (vFSFv) - (SFv)^\nu (vFv)]. \end{aligned} \quad (243)$$

We note that all the potentially divergent terms (two last terms in (242) and in (243)), arising due to the contribution from $\dot{S} \sim \frac{1}{\sqrt{-vGv}}$, disappear on the symmetry grounds. We substitute non vanishing terms into (238) obtaining the expression

$$M^i{}_j a^j = \frac{e\sqrt{-vGv}}{m_r c^2} \left\{ (Fv)^i - \mu(\mu-1)b(FSFv)^i + (\mu-1)^2 a(SFF[\eta + \mu bSF]v)^i - v^i \frac{\mu(\mu-1)b}{2(-vGv)} (vFFSv) \right\}, \quad (244)$$

where the matrix

$$M^i{}_j = \delta^i{}_j + \frac{v^i v^\mu \Omega_{\mu j}}{2(-vGv)}, \quad \text{with} \quad \Omega_{\mu j} = 2\delta_{\mu j} + (\mu-1)b(FS + SF)_{\mu j}, \quad (245)$$

has the inverse

$$\tilde{M}^i{}_j = \delta^i{}_j - \frac{v^i v^\mu \Omega_{\mu j}}{2c^2 - (\mu-1)bv^\mu(FS + SF)_{\mu 0}v^0}, \quad (246)$$

with the property

$$\tilde{M}^i{}_j v^j = v^i \frac{2(-vGv)}{2c^2 - (\mu-1)bv^\mu(FS + SF)_{\mu 0}v^0}. \quad (247)$$

Applying the inverse matrix we obtain the acceleration

$$\begin{aligned} a^i = \frac{e\sqrt{-vGv}}{m_r c^2} \left\{ \tilde{M}^i{}_j [(Fv)^j - \mu(\mu-1)b(FSFv)^j + (\mu-1)^2 a(SFF[\eta + \mu bSF]v)^j] - \right. \\ \left. v^i \frac{\mu(\mu-1)b(vFFSv)}{2c^2 - (\mu-1)bv^\mu(FS + SF)_{\mu 0}v^0} \right\}. \end{aligned} \quad (248)$$

For the particle with non anomalous magnetic moment ($\mu = 1$), the right hand side reduces to the Lorentz force, so the expression in braces is certainly non vanishing in the ultra-relativistic limit. Thus the acceleration vanishes only when $v \rightarrow v_{cr}$, where the critical velocity is determined by the equation $vGv = 0$.

Let us estimate the critical velocity. Using the consequence $(\dot{x}SF\dot{x}) = -b(\mu - 1)(\dot{x}FSSF\dot{x})$ of the Pirani condition, and the expression $S^\mu_\alpha S^{\alpha\nu} = -4[\pi^2\omega^\mu\omega^\nu + \omega^2\pi^\mu\pi^\nu]$, we write

$$-(\dot{x}G\dot{x}) = c^2 - \mathbf{v}^2 + 4b^2(\mu - 1)^2[\pi^2(\omega F\dot{x})^2 + \omega^2(\pi F\dot{x})^2]. \quad (249)$$

As π and ω are space-like vectors (see the discussion below Eq. (70)), the last term is non-negative, so $v_{cr} \geq c$. We show that generally this term is nonvanishing function of velocity, then $v_{cr} > c$. Assume the contrary, that this term vanishes at some velocity, then

$$\begin{aligned} \omega F\dot{x} &= -\omega^0(\mathbf{E}\mathbf{v}) + (\boldsymbol{\omega}, c\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0, \\ \pi F\dot{x} &= -\pi^0(\mathbf{E}\mathbf{v}) + (\boldsymbol{\pi}, c\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0. \end{aligned} \quad (250)$$

This implies $c(\mathbf{DE}) + (\mathbf{D}, \mathbf{v} \times \mathbf{B}) = 0$. Consider the case $\mathbf{B} = 0$, then it should be $(\mathbf{DE}) = 0$. On other hand, for the homogeneous field the quantity $S^{\mu\nu}F_{\mu\nu} = 2[(\mathbf{DE}) + 2(\mathbf{SB})] = 2(\mathbf{DE})$ is a constant of motion. Hence we can take the initial conditions for spin such, that $(\mathbf{DE}) \neq 0$ at any instant, this implies $v_{cr} > c$.

C. Ultra-relativistic limit within the geometry determined by effective metric

According to the previous section, if we insist to preserve the usual special-relativity definitions of time and distance (182), the speed of light does not represent special point of the equation for trajectory. Acceleration of the particle with anomalous magnetic moment generally vanishes at the speed slightly higher than the speed of light. Hence we arrive at a rather surprising result that speed of light does not represent maximum velocity of the manifestly relativistic equation (244). This state of affairs is unsatisfactory because the Lorentz transformations have no sense above c , so two observers with relative velocity $c < v < v_{cr}$ will not be able to compare results of their measurements.

To keep the condition $v_{cr} = c$, we use formal similarity of the matrix G appeared in (230) with space-time metric. Then we can follow the general-relativity prescription of section II to define time and distance in the presence of electromagnetic field. That is we use G of Eq. (230) to define the three-dimensional geometry (5)-(8). The effective metric depends on x^i via the field strength $F(x^0, x^i)$, and on x^0 via the field strength as well as via the spin-tensor $S(x^0)$. So the effective metric is time-dependent even in stationary electromagnetic field. With these definitions we have, by construction, $-\dot{x}G\dot{x} = (\frac{dt}{dx^0})^2(c^2 - (\mathbf{v}\gamma\mathbf{v}))$, so the critical speed coincides with the speed of light. The intervals of time and distance are given now by Eq. (5) and (6), they slightly differ from those in empty space.

In the present case, the expression for three-acceleration can be obtained in closed form in an arbitrary electromagnetic field. We present Eq. (233) in the form (43)

$$DDx^\mu = \mathcal{F}^\mu = -Dx^\mu \frac{Dm_r(S)}{m_r} - T^\mu{}_\nu D\tilde{T}^\nu{}_\alpha(S) Dx^\alpha + T^\mu{}_\nu \left\{ \frac{e}{m_r c^2} (FDx)^\nu + \frac{\mu e}{4m_r^2 c^3} \partial^\nu(SF) + \frac{\mu}{m_r} DZ^\nu \right\}. \quad (251)$$

Then the acceleration is given by (45). The first two terms on right hand side of (251) give potentially divergent contributions arising from the piece $\dot{S} \sim \frac{1}{\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}}$ of Eq. (234). In the previous section we have seen that the dangerous contribution contained in the second term disappears. To analyse the first term, we substitute \mathcal{F}^i from (251) into (45). With use the property $\tilde{M}^i{}_j v^j = v^i \frac{c^2 - \mathbf{v}\gamma\mathbf{v}}{c^2}$, we obtain the acceleration

$$\begin{aligned} a^i &= (c^2 - \mathbf{v}\gamma\mathbf{v}) \left[-v^i \frac{\dot{m}_r}{m_r c^2} - \frac{\tilde{M}^i{}_j T^j{}_\nu \dot{\tilde{T}}^\nu{}_\alpha v^\alpha}{c^2 - \mathbf{v}\gamma\mathbf{v}} + \right. \\ &\quad \left. \tilde{M}^i{}_j T^j{}_\nu \left\{ \frac{e}{m_r c^2 \sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}} (Fv)^\nu + \frac{\mu e}{4m_r^2 c^3} \partial^\nu(SF) + \frac{\mu}{m_r \sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}} \dot{Z}^\nu \right\} \right] + \\ &\quad \tilde{M}^i{}_j \tilde{\Gamma}^j{}_{kl}(\gamma) v^k v^l + \frac{1}{2} \left(\frac{dt}{dx^0} \right)^{-1} \left[(\mathbf{v}\partial_0 \gamma \gamma^{-1})^i - \frac{v^i}{c^2} (\mathbf{v}\partial_0 \gamma \mathbf{v}) \right], \end{aligned} \quad (252)$$

so the divergency due to $\dot{m}_r \sim \frac{1}{\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}}$ is cancelled by the factor in front of this term. In the result, the acceleration is finite as $v \rightarrow c$. Besides, taking into account the property $(\mathbf{v}\gamma)_i \tilde{M}^i{}_j = (\mathbf{v}\gamma)_j \frac{c^2 - \mathbf{v}\gamma\mathbf{v}}{c^2}$, we conclude that the longitudinal acceleration (46)

$$\mathbf{v}\gamma\mathbf{a} = \frac{(c^2 - \mathbf{v}\gamma\mathbf{v})^2}{c^2} (\mathbf{v}\gamma\mathcal{F}) + \frac{c^2 - \mathbf{v}\gamma\mathbf{v}}{c^2} \left[(\mathbf{v}\gamma)_i \tilde{\Gamma}^i{}_{kl}(\gamma) v^k v^l + \frac{1}{2} \left(\frac{dt}{dx^0} \right)^{-1} (\mathbf{v}\partial_0 \gamma \mathbf{v}) \right], \quad (253)$$

vanishes in this limit.

VIII. CONCLUSION

In this work we have studied behavior of ultra-relativistic spinning particle in external fields. To construct interaction of spin with external fields and to analyze its influence on the trajectory of the particle, we used the vector model of spin. Minimal interaction with gravity was formulated starting from the Lagrangian variational problem without auxiliary variables (55). The non-minimal interaction with gravity through the gravimagnetic moment κ [23] has been achieved in the Lagrangian with one auxiliary variable (176).

The variational problems imply the fixed value of spin (78), that is, they correspond to an elementary spin one-half particle. The vector model also allowed us to construct the Lagrangian action (94) with unfixed spin and with a mass-spin trajectory constraint, that is with the properties of Hanson-Regge relativistic top [20]. In this model appeared the fundamental length scale and spin has four physical degrees of freedom. At last, interaction of spinning particle with magnetic moment μ with an arbitrary electromagnetic field was achieved in the Lagrangian action with one auxiliary variable (211). Equations of motion of minimally interacting spinning particle (that is with $\kappa = 0$) have been identified with Mathisson-Papapetrou-Tulczyjew-Dixon equations. They are widely used in the current literature for description of rotating bodies in general relativity. To study the class of trajectories of a body with fixed integrals of motion $\sqrt{-P^2} = k$ and $S^2 = \beta$, we can use our spinning particle with $m = \frac{k}{c}$ and $\alpha = \frac{\beta}{8}$.

To study our general-covariant equations in the laboratory frame, we used the Landau-Lifshitz 1 + 3-splitting formalism of four-dimensional pseudo Riemann space, where the basic structure is a congruence of one-dimensional timelike curves identified with worldlines of the laboratory clocks. This formalism allows one to determine the time interval, distance and then velocity between two infinitesimally closed points x^μ and $x^\mu + \delta x^\mu$ of the particle's worldline. The basic requirement for definition of the three-dimensional quantities is that speed of light should be a coordinate-independent notion. Due to the decomposition of spacetime into time + space, one manipulates only with time-varying vector and tensor fields. In the resulting three-dimensional geometry with Riemannian scalar product, we asked on the notion of a constant vector field. We have suggested the notion (26) which follows from the geometric requirement that scalar product of constant fields does not depend on the point where it was computed. For the vector field of velocity, its deviation from the constant field has given us the acceleration (27). Then we showed that the definition adopted is consistent with the basic principle of general relativity: massive spinless particle, propagating in a gravitational field along a four-dimensional geodesic, can not exceed the speed of light. With this definition at hands, we analysed ultra relativistic behavior of the spinning particle in external fields.

Evolution of the fast MPTD-particle in the laboratory frame was studied on the base of Lagrangian equations (92) and (93). In these equations we observed the emergence of the effective metric (89) instead of the original one. We have examined the two metrics as candidates for construction of three-dimensional space-time geometry (5)-(8) probed by the particle. In both cases the MPTD-equations have unsatisfactory behavior in the ultra-relativistic limit. In particular, the three-dimensional acceleration (27) increases with velocity and becomes infinite in the limit.

Further, we showed that spinning particle with $\kappa = 1$ is free of the problems detected in MPTD-equations. For this value of gravimagnetic moment the effective metric does not appear and the three-dimensional geometry should be defined, unambiguously, with respect to the original metric. Critical velocity of the theory coincides with the speed of light and three-dimensional acceleration vanishes as $v \rightarrow c$. So the spinning particle with gravimagnetic moment $\kappa = 1$ seems more promising candidate for the description of a relativistic rotating body in general relativity. An interesting property of the resulting equations is that spin ceases to affect the trajectory in ultra-relativistic limit: the trajectory of spinning particle becomes more and more close to that of spinless particle as $v \rightarrow c$. Besides, the spin precesses with finite angular velocity in this limit.

Equations in electromagnetic and in gravitational fields become very similar after the identification $\mu \sim \kappa$ and $R_{\mu\nu\alpha\beta}S^{\alpha\beta} \sim F_{\mu\nu}$. In particular, interaction of spin with electromagnetic field in Minkowski space also produces the effective metric (230) for the particle with anomalous magnetic moment $\mu \neq 1$. If we insist on the usual special-relativity notions of time and distance, the critical speed turns out more than the speed of light. To preserve the equality $v_{cr} = c$, we are forced to assume that particle in electromagnetic field probes the three-dimensional geometry determined with respect to the effective metric instead of the Minkowski metric. In the result, we have rather unusual picture of the Universe filled with spinning matter. Since G depends on spin, in this picture there is no unique space-time manifold for the Universe of spinning particles: each particle will probe his own three-dimensional geometry. In principle this could be an observable effect. With the effective metric (230), the equation (5) implies that the time of life of muon in electromagnetic field and in empty space should be different.

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- [1] A. Trautman, *Lectures on general relativity*, Gen. Rel. Grav. **34**, 721 (2002).
 - [2] M. Mathisson, *Neue Mechanik materieller Systeme*, Acta Phys. Polon. **6** (1937) 163; Republication: Gen. Rel. Grav. **42**, 1011 (2010).
 - [3] A. Papapetrou, *Spinning Test-Particles in General Relativity. I*, Proc. Roy. Soc. Lond. **A 209** (1951) 248.
 - [4] W. M. Tulczyjew, *Motion of multipole particles in general relativity theory binaries*, Acta Phys. Polon. **18** (1959) 393.
 - [5] W. G. Dixon, *A Covariant Multipole Formalism for Extended Test Bodies in General Relativity*, Nuovo Cimento **34** (1964) 317.
 - [6] W. G. Dixon, *On a Classical Theory of Charged Particles with Spin and the Classical Limit of the Dirac Equation*, Nuovo Cimento **38** (1965) 1616.
 - [7] F. A. E. Pirani, Acta. Phys. Pol. **15** (1956) 389.
 - [8] L. H. Thomas, *The kinematics of an electron with an axis*, Philosophical Magazine and Journal of Science **3** S.7, No.13 (1927) 1.
 - [9] J. Frenkel, *Die elektrodynamik des rotierenden elektrons*, Z. Phys **37** (1926) 243.
 - [10] J. Frenkel, *Spinning Electrons*, Nature **117** (1926) 653.
 - [11] V. Bargmann, L. Michel, and V. L. Telegdi, Phys. Rev. Lett. **2** (1959) 435.
 - [12] A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (MacMillan, New York 1964).
 - [13] H. C. Corben, *Classical and quantum theories of spinning particles*, Holden-Day, San Francisco, 1968.
 - [14] I. B. Khriplovich and A. A. Pomeransky, “Equations of motion of spinning relativistic particle in external fields,” J. Exp. Theor. Phys. **86**, 839 (1998) [Zh. Eksp. Teor. Fiz. **113**, 1537 (1998)] [gr-qc/9710098].
 - [15] C. M. Will, *Living Reviews Relativity*, bf 17 (2014) 4.
 - [16] A. B. Balakin and V. A. Popov, *Spin-axion coupling*, arXiv:1509.06058 (gr-qc).
 - [17] A. A. Deriglazov and A. M. Pupasov-Maksimov, *Lagrangian for Frenkel electron and position’s non-commutativity due to spin*, Eur. Phys. J. C **74**, 3101 (2014) [arXiv:1312.6247 [hep-th]].
 - [18] M. Mukunda, H. van Dam and L. C. Biedenharn, *Relativistic models of extended hadrons obeying a mass-spin trajectory constraint*, Lecture Notes in Physics **165** Springer-Verlag, 1982.
 - [19] A. M. Frydryszak, *Lagrangian models of the particles with spin: the first seventy years*, hep-th/9601020.
 - [20] A. J. Hanson and T. Regge, *The relativistic spherical top*, Annals of Physics **87**(2) (1974) 498.
 - [21] S. Hojman, *Lagrangian theory of motion of spinning particles in torsion gravitational theories*, Phys. Rev. **18** (1978) 8.
 - [22] C. Armaza, M. Banados and Benjamin Koch, *Can Schwarzschild black holes be accelerators of spinning massive particles?* arXiv:1510.01223 (gr-qc).
 - [23] I. B. Khriplovich, *Particle with internal angular momentum in a gravitational field*, Sov. Phys. JETP **69**, 217 (1989) [Zh. Eksp. Teor. Fiz. **96**, 385 (1989)].
 - [24] J. W. van Holten, *On the electrodynamics of spinning particles*, Nucl. Phys. **B356** (1991) 3.
 - [25] L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields, Course of Theoretical Physics, V. 2*, (Pergamon Press, third ed., 2011).
 - [26] E.ourgoulhon, *3+1 Formalism in General Relativity*, Lecture Notes in Physics **846** (2012) 1.
 - [27] W. Guzmán Ramírez, A. A. Deriglazov and A. M. Pupasov-Maksimov, *Frenkel electron and a spinning body in a curved background*, JHEP **1403**, 109 (2014) [arXiv:1311.5743 [hep-th]].
 - [28] A. A. Deriglazov and A. M. Pupasov-Maksimov, *Frenkel electron on an arbitrary electromagnetic background and magnetic Zitterbewegung*, Nucl. Phys. B **885**, 1 (2014) [arXiv:1401.7641 [hep-th]].
 - [29] A. A. Deriglazov, *Lagrangian for the Frenkel electron*, Phys. Lett. B **736**, 278 (2014) [arXiv:1406.6715 [physics.gen-ph]].
 - [30] W. G. Ramirez and A. A. Deriglazov, *Lagrangian formulation for Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations*, arXiv:1509.04926 (gr-qc).
 - [31] A. A. Deriglazov and W. G. Ramirez, *Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations in ultra-relativistic regime and gravimagnetic moment*, arXiv:1509.05357 (gr-qc).
 - [32] A. A. Deriglazov and W. G. Ramirez, *World-line geometry probed by fast spinning particle*, Mod. Phys. Lett. **A 30** (2015) 1550101; arXiv:1409.4756 (physics.gen-ph).
 - [33] A. Staruszkiewicz, *Fundamental relativistic rotator*, Acta Physica Polonica B Proceedings Supplement **1** (2008) 109.
 - [34] A. A. Deriglazov and A. M. Pupasov-Maksimov, *Geometric Constructions Underlying Relativistic Description of Spin on the Base of Non-Grassmann Vector-Like Variable*, SIGMA **10**, 012 (2014) [arXiv:1311.7005 [math-ph]].
 - [35] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964).
 - [36] D. M. Gitman and I. V. Tyutin, *Quantization of fields with constraints* (Springer-Verlag, Berlin, 1990).
 - [37] A. Deriglazov, *Classical Mechanics: Hamiltonian and Lagrangian Formalism* (Springer-Verlag, 2010).
 - [38] K. S. Thorne and J. B. Hartle, *Laws of motion and precession for black holes and other bodies*, Phys. Rev. **D 31** (1985) 1815.